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CH: 5

FRICTION

Smooth Body

If the force of friction between two bodies in contact is zero, then the bodies are called smooth bodies.

Rough Body

If the force of friction between two bodies in contact is not zero, then bodies are called rough bodies.

Light Body

An ideal body whose weight can be neglected is called a light body.

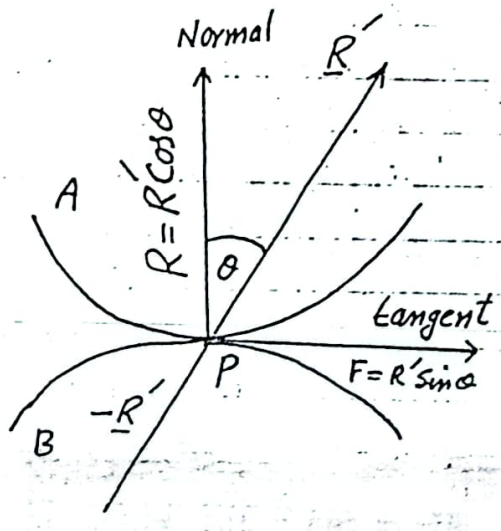
Note # In nature no perfectly smooth body exists. Each body offers some resistance although it may be very small as in case of glass, steel etc.

Static Equilibrium

Equilibrium of a body or a system of bodies in the state of rest is called static equilibrium.

Frictional Force

In case of smooth bodies in contact with each other the action and reaction are along the common normal at the point of contact of bodies. But there is no perfectly smooth body since



in nature, therefore action and reaction are inclined at some angle to the common normal at the point of contact of bodies.

Suppose two bodies A & B are in Contact with each other at point P. These bodies have a common tangent plane at point P and hence a common normal at point P. Let θ be the angle of reaction $\underline{R}'$ on body A with the common normal or normal reaction. Then component of $\underline{R}'$ along the common tangent at pt P is $F = R'\sin\theta$ and the component of $\underline{R}'$ along the common normal is $R = R'\cos\theta$. The component $F = R'\sin\theta$ in the tangent plane and along the common tangent prevents the sliding or motion of A on B and is called frictional force or force of friction or simply friction. The component $R = R'\cos\theta$ prevents the penetration of A into B and is called normal reaction or normal pressure.

Results # 1) # If $F = R'\sin\theta = 0$, then $\theta = 0$ i.e reaction $\underline{R}'$ will act along the normal and the contact is smooth. This is an ideal case of no friction.

2) # If $F = R'\sin\theta \neq 0$, then $\theta \neq 0$ and the contact is rough. So

$$F = R'\sin\theta \qquad R = R'\cos\theta$$

$$\Rightarrow F/R = \tan\theta$$

Since at a given point of contact, R is supposed constant, value of F depends upon θ. If θ increases from 0, then and F also increases and when θ attains its maximum acute value λ, the frictional force, $R\tan\lambda$ is maximum and is called limiting friction. The angle λ is called angle of friction.

3) # At a rough Contact

force of friction $\leq$ Limiting friction

or $\qquad R'\tan\theta \leq R'\tan\lambda$

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or $\tan\theta \leq \tan\lambda$

or $\theta \leq \lambda$

The equilibrium in case of limiting friction is called limiting equilibrium.

If θ becomes slightly greater than λ, then equilibrium will be destroyed and the body begins to slide on the other. The friction in this case will be slightly less than $R\tan\lambda$, the maximum friction and is called dynamical or kinetic or sliding friction.

Types of Friction

1# Static or Statical friction

If the bodies in contant are at rest, the friction is called static friction.

2# Kinetic or Dynamical or Sliding friction#

If one body moves or slides along the other body, then friction is called kinetic or sliding or dynamical friction.

3# Non-Limiting friction#

When two bodies are in contact with eachother but neither of them is on the point of sliding on the other, then frictional force is called non-limiting friction.

4# Limiting Friction#

When two bodies are in contact with each other and one body is just on the point sliding but actual motion has not started, then friction present is called limiting friction. This is the maximum static friction which a body can offer to another body before the motion of body. Equilibrium in this case is called limiting equilibrim.

Note# Static and dynamical frictions are less than limiting friction or maximum friction.

Laws of Friction

The Laws of both static and dynamical friction are stated as

1) # The direction of friction is always opposite to the direction in which the body tends to move or moves.

2) # The magnitude of friction is, up to a certain, limit, equal to the the force tending to produce motion.

3) # Only a certain amount of friction can be called into play and in each particular case it cannot exceed a certain limit.

4) # The magnitude of the limiting friction (for given surfaces) bears a constant ratio μ to the normal reaction between the surfaces. The ratio μ depends upon the nature of the surfaces and is called the co-efficient of friction ; i.e

$$F/R = \mu \quad \text{or} \quad F = \mu R$$

where F is the limiting friction and R is the normal reaction.

5) # The amount of friction is independent of the areas and shape of the surfaces in contact provided the normal reaction between the surfaces remains constant.

6) # Dynamical friction is independent of the velocity and is proportional to the normal reaction. but is slightly less than the limiting friction.

Coefficient of Kinetic friction

In case of kinetic friction F, we have

$$F/R = \mu'$$

where μ' is a constant and is called co-efficient of kinetic friction.

Note # In case of the same substances μ' is slightly less than μ, the co-efficient of friction.

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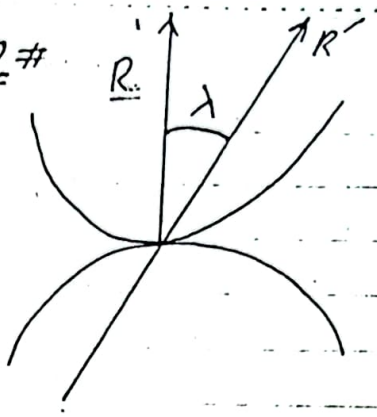
Angle of Friction



When two bodies in contact are in limiting friction, the angle between the resultant mutual reaction $\underline{R}'$ and the normal reaction $\underline{R}$ is called angle of friction. It is denoted by λ.

$$R = R' \cos\lambda \rightarrow (1)$$
$$F = R' \sin\lambda \rightarrow (2)$$

Dividing (2) by (1)

$$\tan\lambda = F/R = \mu$$
$$\Rightarrow \lambda = \tan^{-1}(\mu)$$

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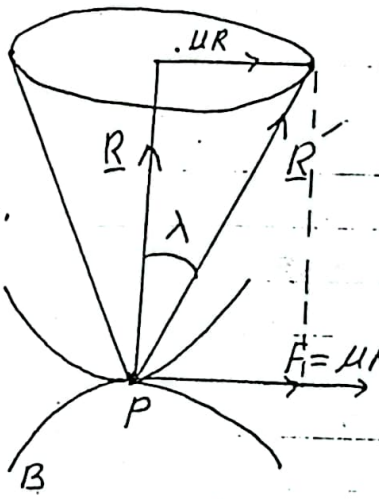
Cone of Friction

A cone with vertex at the point of contact, axis along the common normal and semi-vertical angle equal to the angle of friction is called cone of friction.

This cone is formed by drawing all lines inclined to the normal at an angle λ i.e generators of cone are inclined at ang λ with R



Remarks

The angle θ between the resultant mutual reaction $\underline{R}'$ and the normal reaction $\underline{R}$ at the point of contact of two rough bodies is such that

$$\theta \leq \lambda$$

where λ is the angle of friction.

This shows that $\underline{R}'$ always lies within or on the cone of friction. The line of action of R' is the generator of cone.

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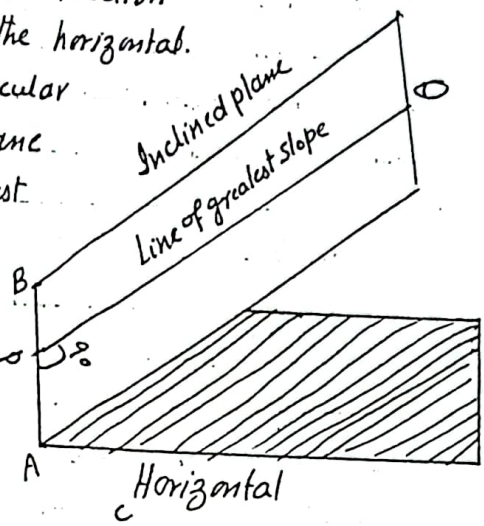
Line of Greatest Slope

Let AB be the line of intersection of an inclined plane with the horizontal. Then any line PQ perpendicular to AB lying in the inclined plane is called the line of the greatest slope of the inclined plane. A particle on an inclined plane in equilibrium / limiting equilibrium always lies on the line of the greatest slope.



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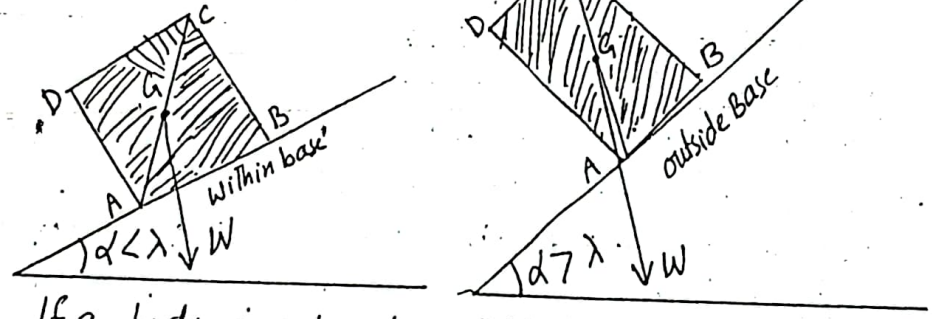
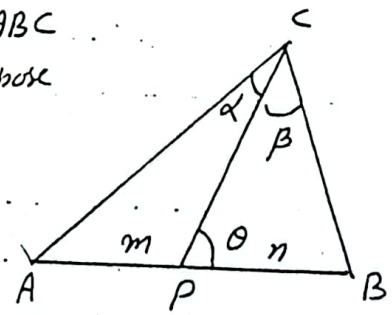


If a body is placed with its base in contact with an inclined plane, it will stand or fall according as the vertical line drawn through its C.G meets the plane within or outside the base of the body in contact with the plane.

(m, n) Theorem

Let ABC be a triangle as shown in fig. Suppose P divides AB in ratio m : n and ∠ACP = α, ∠BCP = β



Then we have

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot A - m\cot B$$

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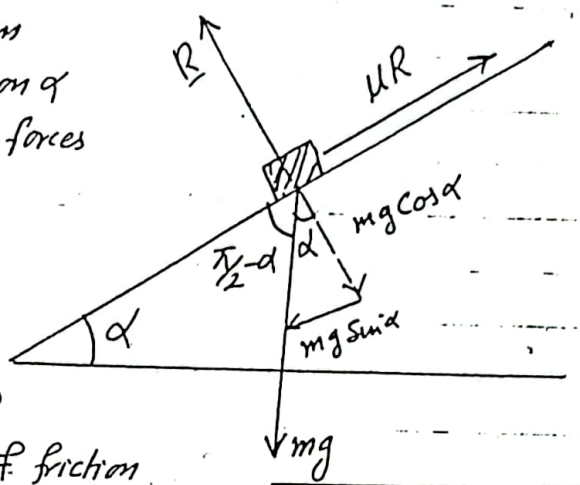
Condition for limiting Equilibrium on a rough Inclined Plane under no external force

Let a particle of mass m be in limiting equilibrium on a plane of inclination α under its own weight. The forces acting on the particle are

(1) its weight mg acting vertically downward

(2) the normal reaction R

(3) the force of friction μR



Let λ be the angle of friction and μ be the coefficient of friction.

Resolving the forces along and perpendicular to the inclined plane

$$mg\cos\alpha = R \longrightarrow (1)$$
$$mg\sin\alpha = \mu R \longrightarrow (2)$$

dividing (2) by (1)

$$\tan\alpha = \mu = \tan\lambda$$
$$\Rightarrow \alpha = \lambda$$

This shows that when a particle is in limiting equilibrium on a rough inclined plane under its own weight, the inclination of the plane will be equal to the angle of friction.

Angle between two lines is same as the angle b/w their perpendiculars. Therefore angle b/w mg and normal to plane is α

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Alternate Method

Since the particle is in limiting equilibrium under the action of three forces mg, R and μR, therefore forces must be coplanar and also the Lami's theorem is applicable.

Angle b/w R and mg $= \pi/2 + \pi/2 - \alpha = \pi - \alpha$

By Lamis Theorem

$$\frac{\mu R}{\sin(\pi-\alpha)} = \frac{R}{\sin(\frac{\pi}{2}+\alpha)}$$

$$\frac{\mu}{\sin\alpha} = \frac{1}{\cos\alpha}$$

$$\Rightarrow \tan\alpha = \mu = \tan\lambda$$

$$\Rightarrow \alpha = \lambda$$

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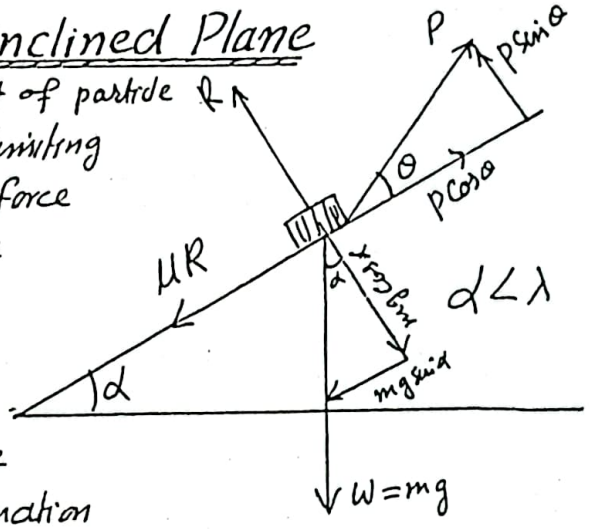
To find the least force to drag a particle up a rough inclined Plane

Let mg be the weight of particle and the particle is in limiting equilibrium due to a force P applied at particle making an angle θ with the plane. Due to force P the particle is just to slide up. Let α be the inclination of the plane.



Resolving the forces along and perpendicular to the plane.

$$mg\sin\alpha + \mu R = P\cos\theta \longrightarrow (1)$$

$$mg\cos\alpha = R + P\sin\theta \longrightarrow (2)$$

From (2)

$$R = mg\cos\alpha - P\sin\theta$$

putting in (1)

$$mg\sin\alpha + \mu(mg\cos\alpha - P\sin\theta) = P\cos\theta$$

$$P\cos\theta + \mu P\sin\theta = mg(\sin\alpha + \mu\cos\alpha)$$

$$P(\cos\theta + \mu\sin\theta) = mg(\sin\alpha + \mu\cos\alpha)$$

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$$P = \frac{mg(\sin\alpha + \mu\cos\alpha)}{\cos\theta + \mu\sin\theta}$$

$$= \frac{mg(\sin\alpha + \tan\lambda\cos\alpha)}{\cos\theta + \tan\lambda\sin\theta} \quad (\mu = \tan\lambda)$$

$$= \frac{mg\left(\sin\alpha + \frac{\sin\lambda}{\cos\lambda}\cos\alpha\right)}{\cos\theta + \frac{\sin\lambda}{\cos\lambda}\sin\theta}$$

$$= \frac{mg(\sin\alpha\cos\lambda + \sin\lambda\cos\alpha)}{\cos\theta\cos\lambda + \sin\lambda\sin\theta}$$

$$= mg\,\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$$

P will be least when $\cos(\theta-\lambda)$ is maximum. i.e

$$\cos(\theta-\lambda) = 1$$
$$\Rightarrow \quad \theta - \lambda = 0$$
$$\Rightarrow \quad \theta = \lambda$$

Therefore

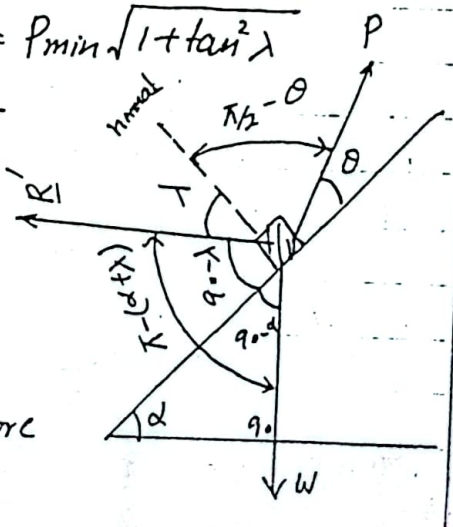
$$P_{min} = mg\sin(\alpha+\lambda)$$

If $\theta = 0$ i.e if P acts parallel to the plane, then

$$P = mg\,\frac{\sin(\alpha+\lambda)}{\cos\lambda} = mg\sin(\alpha+\lambda)\sec\lambda$$

$$= P_{min}\sec\lambda = P_{min}\sqrt{1+\tan^2\lambda}$$

$$= P_{min}\sqrt{1+\mu^2}$$



Alternate Method

Since the particle is in equilibrium under the action of three forces P, mg, R', therefore by Lami's theorem

$$\frac{P}{\sin[\pi-(\alpha+\lambda)]} = \frac{mg}{\sin\{\frac{\pi}{2}-(\theta-\lambda)\}}$$

$$\frac{P}{\sin(\alpha+\lambda)} = \frac{mg}{\cos(\theta-\lambda)}$$

$$P = \frac{mg\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}$$

P is least when $\cos(\theta-\lambda)$ is max i.e

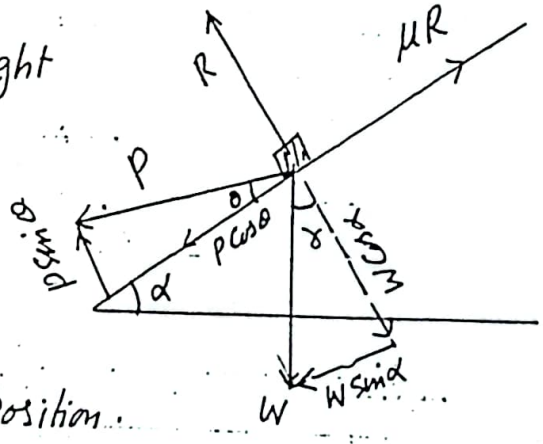
$$\cos(\theta-\lambda) = 1$$
$$\Rightarrow \theta-\lambda = 0$$
$$\Rightarrow \theta = \lambda$$
$$P_{min} = mg\sin(\alpha+\lambda)$$

If $\theta = 0$ i.e if P is applied parallel to the plane, then

$$P = mg\frac{\sin(\alpha+\lambda)}{\cos\lambda} = mg\sin(\alpha+\lambda)\sec\lambda$$
$$= P_{min}\sqrt{1+\tan^2\lambda} = P_{min}\sqrt{1+\mu^2}$$

To find the least force to drag a Particle down a rough inclined Plane

Let $W=mg$ be the weight of particle and P be the force acting at an angle θ with the inclined plane of inclination α. Due to force P the particle is just in sliding down position.



Resolving the forces along and perpendicular to the plane

$$\mu R = P\cos\theta + W\sin\alpha \quad \rightarrow (1)$$

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$$R + P\sin\theta = W\cos\alpha \quad \longrightarrow (2)$$

From (2):

$$R = W\cos\alpha - P\sin\theta$$

Putting in (1)

$$\mu(W\cos\alpha - P\sin\theta) = P\cos\theta + W\sin\alpha$$

$$\mu P\sin\theta + P\cos\theta = \mu W\cos\alpha - W\sin\alpha$$

$$P(\mu\sin\theta + \cos\theta) = W(\mu\cos\alpha - \sin\alpha)$$

$$P = \frac{W(\mu\cos\alpha - \sin\alpha)}{\mu\sin\theta + \cos\theta}$$

$$= \frac{W(\tan\lambda\cos\alpha - \sin\alpha)}{\tan\lambda\sin\theta + \cos\theta} \quad (\mu = \tan\lambda)$$

$$= \frac{W(\sin\lambda\cos\alpha - \cos\lambda\sin\alpha)}{\sin\lambda\sin\theta + \cos\lambda\cos\theta}$$

$$= W\frac{\sin(\lambda-\alpha)}{\cos(\theta-\lambda)} \quad (\because \alpha < \lambda)$$

P will be least when $\cos(\theta-\lambda)$ is maximum i.e

$$\cos(\theta-\lambda) = 1$$

$$\Rightarrow \theta - \lambda = 0$$

$$\Rightarrow \theta = \lambda$$

Therefore

$$P_{min} = W\sin(\lambda-\alpha)$$

If $\theta = 0$ i.e if P acts parallel to the plane, then

$$P = W\frac{\sin(\lambda-\alpha)}{\cos\lambda} = W\sin(\lambda-\alpha)\sec\lambda$$

$$= P_{min}\sqrt{1+\tan^2\lambda}$$

$$= P_{min}\sqrt{1+\mu^2}$$

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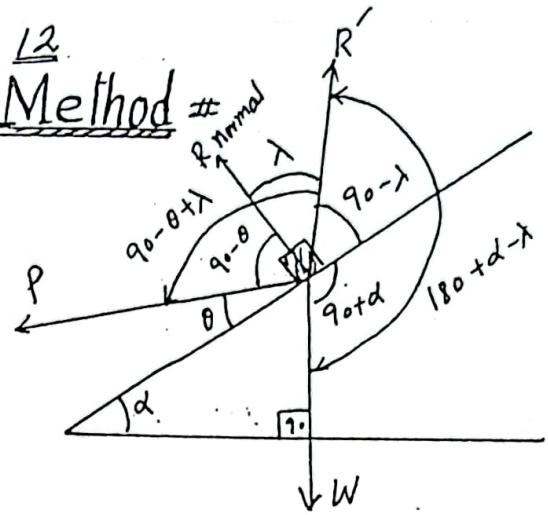
Alternate Method

Angles between forces P, W, R′ as shown. Since the particle is in equilibrium under the action of three forces, therefore by Lamis Theorem, we have



$$\frac{P}{\sin[180+\alpha-\lambda)} = \frac{W}{\sin(90-\theta+\lambda)}$$

$$\frac{P}{\sin\{180-(\lambda-\alpha)\}} = \frac{W}{\sin[90-(\theta-\lambda)]}$$

$$\frac{P}{\sin(\lambda-\alpha)} = \frac{W}{\cos(\theta-\lambda)}$$

$$\Rightarrow P = W\frac{\sin(\lambda-\alpha)}{\cos(\theta-\lambda)}$$

Next treatment is same as above.

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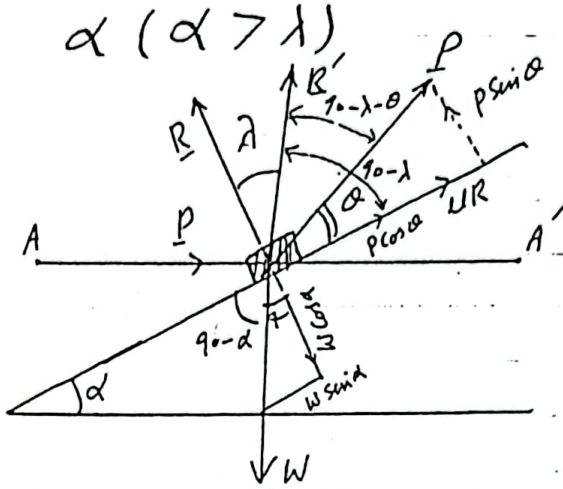
Note # A particle lying on a rough inclined plane of inclination α is in equilibrium or limiting equilibrium if

$$\alpha \leq \lambda$$

When α becomes slightly greater than λ, the equilibrium will be destroyed and the particle will start sliding down the plane. In this case some external force must be required to maintain the equilibrium.

Next we discuss this fact.

To find the least force necessary to Just support a heavy Particle on an inclined Plane of inclination α ($\alpha > \lambda$)



If $\alpha > \lambda$, then the body will slide down under its own weight and friction will not be sufficient to prevent sliding. So a force will be required to support or the body or to prevent the motion. Let P be the force applied at an angle θ with the plane and is just necessary for limiting equilibrium.

Resolving the forces along and perpendicular to the plane.

$$W\sin\alpha = P\cos\theta + \mu R \longrightarrow (1)$$

$$P\sin\theta + R = W\cos\alpha \longrightarrow (2)$$

From (2)

$$R = W\cos\alpha - P\sin\theta$$

putting in (1)

$$W\sin\alpha = P\cos\theta + \mu(W\cos\alpha - P\sin\theta)$$

$$W\sin\alpha - \mu W\cos\alpha = P\cos\theta - \mu P\sin\theta$$

$$P = \frac{W(\sin\alpha - \mu\cos\alpha)}{\cos\theta - \mu\sin\theta}$$

$$= W\frac{\sin\alpha - \tan\lambda\cos\alpha}{\cos\theta - \tan\lambda\sin\theta}$$

$$P = \frac{W(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda)}{\cos\theta\cos\lambda - \sin\lambda\sin\theta}$$

$$= W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

P will be least when $\cos(\theta+\lambda)$ is max. i.e

$$\cos(\theta+\lambda) = 0$$
$$\theta+\lambda = 0$$
$$\theta = -\lambda$$

This shows that P acts along $\overrightarrow{AA'}$

Therefore

$$P_{min} = W\sin(\alpha-\lambda)$$

Alternate Method

Angle between R' and $W = \frac{\pi}{2} + \lambda + \frac{\pi}{2} - \alpha$

$$= \pi - (\alpha-\lambda)$$

Angle between R' and $P = \pi/2 - \lambda - \theta$

$$= \frac{\pi}{2} - (\theta+\lambda)$$

Since the particle is in equilibrium under the action of three forces, therefore by Lami's theorem

$$\frac{P}{\sin\{\pi-(\alpha-\lambda)\}} = \frac{W}{\sin\{\frac{\pi}{2}-(\theta+\lambda)\}}$$

$$\frac{P}{\sin(\alpha-\lambda)} = \frac{W}{\cos(\theta+\lambda)}$$

$$P = \frac{W\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

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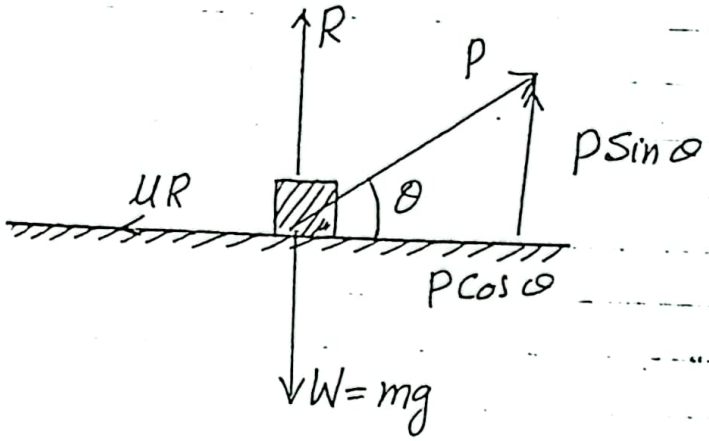
To find the least force required to drag a heavy Particle on a rough Horizontal Plane.



Let a particle of weight W be on the point moving under the action of force P acting at an angle θ with the horizontal plane.

Resolving the forces along and perpendicular to the plane, we have

$$R + P\sin\theta = W \longrightarrow (1)$$

$$P\cos\theta = \mu R \longrightarrow (2)$$

From (1)

$$R = W - P\sin\theta$$

using in (2)

$$P\cos\theta = \mu(W - P\sin\theta)$$

$$P\cos\theta + \mu P\sin\theta = \mu W$$

$$P(\cos\theta + \mu\sin\theta) = \mu W$$

$$P = \frac{\mu W}{\cos\theta + \mu\sin\theta}$$

$$P = \frac{W\tan\lambda}{\cos\theta + \tan\lambda\sin\theta}$$

$$= \frac{W\sin\lambda/\cos\lambda}{\cos\theta + (\sin\lambda/\cos\lambda)\sin\theta}$$

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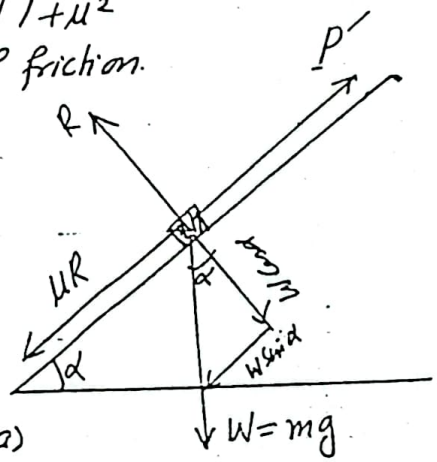
$$P = \frac{W \sin\lambda}{\cos(\theta - \lambda)}$$

P is least if $\cos(\theta-\lambda)$ is maximum i.e

$$\cos(\theta-\lambda) = 1$$
$$\Rightarrow \quad \theta - \lambda = 0$$
$$\Rightarrow \quad \theta = \lambda$$

Thus $P_{min} = W \sin\lambda$

Example # The least force which will move a weight up an inclined plane is of magnitude P. Show that the least force acting parallel to the plane, which will move the weight upwards is $P\sqrt{1+\mu^2}$ where μ is the co-efficient of frichion.



Sol # Let α be the inclination of the plane. The least value of P when inclined at an angle θ with the inclined plane is

$$P = W \sin(\alpha+\lambda) \quad \rightarrow (a)$$

Let P' be the magnitude of the least force which when acting parallel to the plane sets the weight in limiting equilibrium. Then

$$P' = W\sin\alpha + \mu R \quad \rightarrow (1)$$
$$R = W\cos\alpha$$

Thus $P' = W\sin\alpha + \mu W\cos\alpha$

$$= W(\sin\alpha + \mu\cos\alpha)$$

$$= W\cos\alpha\left(\frac{\sin\alpha}{\cos\alpha} + \mu\right)$$

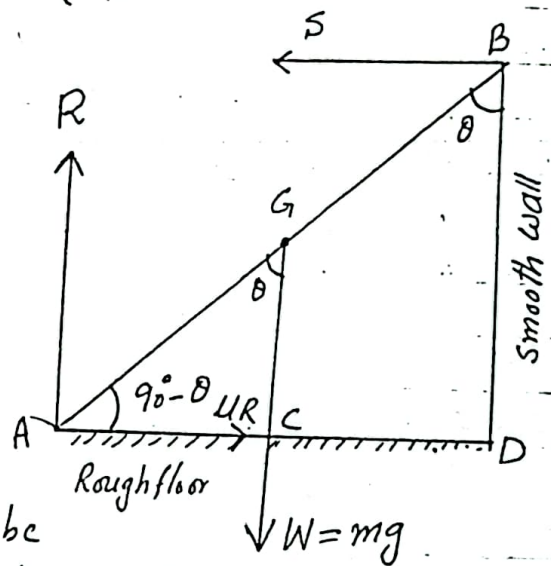
$$= W\cos\alpha\,(\tan\alpha + \tan\lambda)$$

$$P' = \frac{P}{\cos\lambda} \qquad \text{by (a)}$$

$$= P\sec\lambda = P\sqrt{1+\tan^2\lambda}.$$

$$P' = P\sqrt{1+\mu^2}$$

Example # A uniform ladder rests in limiting equilibrium with one end on rough floor whose co-efficient of friction is μ and with the other against a smooth vertical wall. Show that its inclination to the vertical is $\tan^{-1}(\mu)$



Sol # Let $AB = 2l$ be the ladder. Since wall is smooth, therfore reaction of wall is only normal reaction S. R is normal reaction of floor and μR is force of friction. Let α be the inclination of ladder with wall.

All forces are co-planar and in equilibrium.

Resolving the forces horizontally and vertically.

$\because$ the forces are in equilibrium

$\therefore$ horizontal and vertically components balance each other and we have

$$S = \mu R \longrightarrow (1)$$

$$R = W \longrightarrow (2)$$

Putting (2) in (1)

$$S = \mu W \longrightarrow (3)$$

Taking moments about point A.

$$W \cdot AC = S \cdot BD$$

$$W \cdot l \sin\theta = S \cdot 2l \cos\theta$$

$$W \sin\theta = 2S \cos\theta$$

Putting $S = \mu W$ from (3)

$$W \sin\theta = 2\mu W \cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = 2\mu$$

$$\tan\theta = 2\mu$$

$\theta = \tan^{-1}(2\mu)$ is the required inclination of ladder with wall.

Note # Weigh of the ladder acts through the c.g. of rod vertically down and since rod is uniform, therefore its c.g is its middle point.

* We take moments of forces about such a point through which there pass maximum no of forces so that moments of the maximum numbers (may) of forces may be zero

* Eliminate the normal reaction at the point about which moments will be calculated.

Example # A uniform ladder, of length 70 feet, rests against a vertical wall with which it makes an angle of 45°, the co-efficient of friction between the ladder and wall and the respectively being $\frac{1}{3}, \frac{1}{2}$. If a man, whose weight is one half of that of the ladder, ascends the ladder, where will he be when the ladder slips?

Sol # Since the ladder is uniform, therefore its c.g is the middle point and whole weight of the ladder acts through middle point vertically downwards. Let W be the weight of the ladder and R, S be the normal

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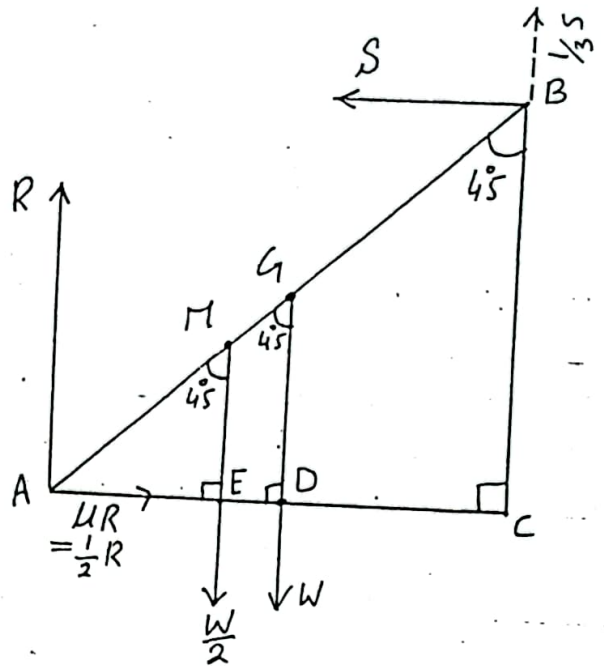

reactions of ground and wall respectively. Suppose a man of wt. $W/2$ ascends upt point M and the ladder becomes at the point slipping. Let $AB = 70$ feet be the ladder. Then

$$AG = 35 \text{ feet.}$$

Also lines of action of W, $W/2$ are parallel to BC and AB is a transversal that cuts these.

Therefore

$$\angle ABC = \angle AGD = \angle AME = 45^\circ$$

(Corresponding angles on parallel lines)

Resolving the forces horizontally and vertically

$$S = \tfrac{1}{2} R \longrightarrow (1)$$

$$R + \frac{1}{3} S = \frac{W}{2} + W = \frac{3W}{2} \longrightarrow (2)$$

from (1) $R = 2S$

putting in (2)

$$2S + \frac{1}{3} S = \frac{3W}{2}$$

$$\tfrac{7}{3} S = \frac{3W}{2}$$

$$\boxed{S = \frac{9}{14}W} \longrightarrow (3)$$

Taking moments about point A.

$$S \cdot BC + \frac{1}{3} S \cdot AC = W \cdot AD + \frac{W}{2} AE$$

$$S \cdot 70\cos 45^\circ + \frac{1}{3} S \cdot 70 \sin 45^\circ = W \cdot 35 \sin 45^\circ + \frac{W}{2} AM \sin 45^\circ$$

$$70S \cdot \frac{1}{\sqrt{2}} + \frac{1}{3} S \cdot 70 \cdot \frac{1}{\sqrt{2}} = W \cdot 35 \cdot \frac{1}{\sqrt{2}} + \frac{W}{2} AM \cdot \frac{1}{\sqrt{2}}$$

Dividing by $\frac{1}{\sqrt{2}}$

$$70S + \frac{70}{3} S = 35W + \frac{AM}{2} W$$

$$\frac{280}{3} S = W\left(35 + \frac{AM}{2}\right)$$

putting $S = \frac{9}{14} W$ from (3)

$$\frac{\overset{20}{\cancel{280}}}{3} \cdot \frac{3}{\underset{7}{\cancel{14}}} \cdot 9W = W\left(35 + \frac{AM}{2}\right)$$

$$60 = 35 + \frac{AM}{2}$$

$$25 = \frac{AM}{2}$$

$$AM = 50 \text{ ft.}$$

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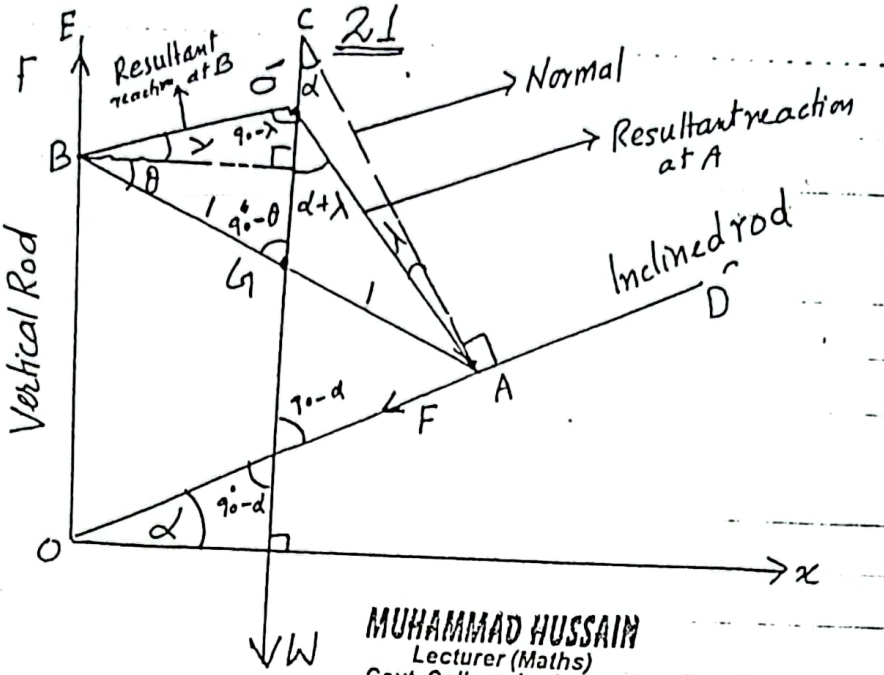
Example # A uniform (ladder) rod slides with its ends on two equally rough rods, one being vertical and the other inclined at an angle α to the horizontal. Show that the angle θ to the horizontal of the moveable rod, when it is on the point of sliding is given by

$$\tan\theta = \frac{1 - 2\mu\tan\alpha - \mu^2}{2(\mu - \tan\alpha)}$$



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Sol # Let OD be inclined rod and OE be the vertical rod. Let AB be the moveable rod of weigh W. The rod AB is acted upon by three forces, the reactions of the fixed rods and its own weight. Let the line of action of W and ⊥ars. to the lower rod meet at a point C and reactions of the fixed rods meet the line of action of W at O' because rod is in equilibrium under the action of three forces. Let θ be the angle of rod AB with horizontal.

From fig

$m\angle O'AC = \lambda$

$m\angle ACO' = \alpha$ (Angle b/w ⊥ars of horizontal and rod OD)

$m\angle BO'G = 90 - \lambda$

$m\angle AO'G = \alpha + \lambda$ (External angle of $\Delta ACO'$ ∵ External angle of triangle is sum of opposite internal angle)

By (m, m) theorem to $\Delta AO'B$

$(1+1)\cot(90^\circ - \theta) = 1\cdot\cot(\alpha+\lambda) - 1\cdot\cot(90^\circ - \lambda)$

$= \frac{1}{\tan(\alpha+\lambda)} - \tan\lambda$

$$2\tan\theta = \frac{1}{\left(\frac{\tan\alpha+\tan\lambda}{1-\tan\alpha\tan\lambda}\right)} - \tan\lambda$$

$$= \frac{1-\tan\alpha\tan\lambda}{\tan\alpha+\tan\lambda} - \tan\lambda$$

$$2\tan\theta = \frac{1-\mu\tan\alpha}{\tan\alpha+\mu} - \mu \quad (\because \tan\lambda=\mu)$$

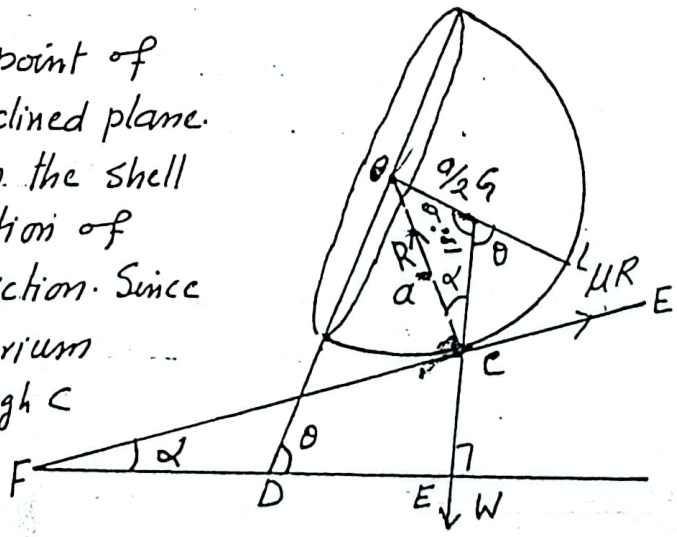
$$= \frac{1-\mu\tan\alpha-\mu\tan\alpha-\mu^2}{\tan\alpha+\mu}$$

$$\text{Tan}\theta = \frac{1-2\mu\tan\alpha-\mu^2}{2(\tan\alpha+\mu)}$$

$$\theta = \text{Tan}^{-1}\left(\frac{1-2\mu\tan\alpha-\mu^2}{2(\tan\alpha+\mu)}\right)$$

Example # A hemi-spherical shell rests on a rough inclined plane whose angle of friction is λ. Show that the inclination of the plane base to the horizontal can not be greater than $\sin^{-1}(2\sin\lambda)$

Sol # Let C be point of Contact with the inclined plane. The forces acting on the shell are its weight, reaction of the plane and the fiction. Since the shell is in equilibrium and R, μR pass through C, therefore line of action



W must also pass through C

If O is the centre of the shell and 'a' is its radius, then C.g G is at distance $OG = \frac{a}{2}$ on symmetry radius. Let θ be the angle of the plane base with horizontal and α be the inclination of the plane.

$m\angle OCG = \alpha$ ($\because$ OC $\perp$ar FE and line of action of W is $\perp$ar to horizontal)

$m\angle CGL = \theta$ (Angle between $\perp$ars of plane base of shell and horizontal)

$\Rightarrow m\angle OGC = 180^\circ - \theta$

Applying the Law of sines to $\triangle OGC$, we have

$$\frac{OG}{\sin\alpha} = \frac{OC}{\sin(\pi-\theta)}$$

$$\frac{a/2}{\sin\alpha} = \frac{a}{\sin\theta}$$

$$\frac{1}{2\sin\alpha} = \frac{1}{\sin\theta}$$

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$$\Rightarrow \sin\theta = 2\sin\alpha$$

$$\Rightarrow \theta = \sin^{-1}(2\sin\alpha)$$

But for equilibrium or limiting equilibrium of a body on inclined plane we have

$$\alpha \leq \lambda$$

$$\Rightarrow \sin\alpha \leq \sin\lambda$$

$$\Rightarrow 2\sin\alpha \leq 2\sin\lambda$$

$$\Rightarrow \sin^{-1}(2\sin\alpha) \leq \sin^{-1}(2\sin\lambda)$$

$$\Rightarrow \theta \leq \sin^{-1}(2\sin\lambda)$$

$\Rightarrow \theta$ cannot be greater than $\sin^{-1}(2\sin\lambda)$

OR

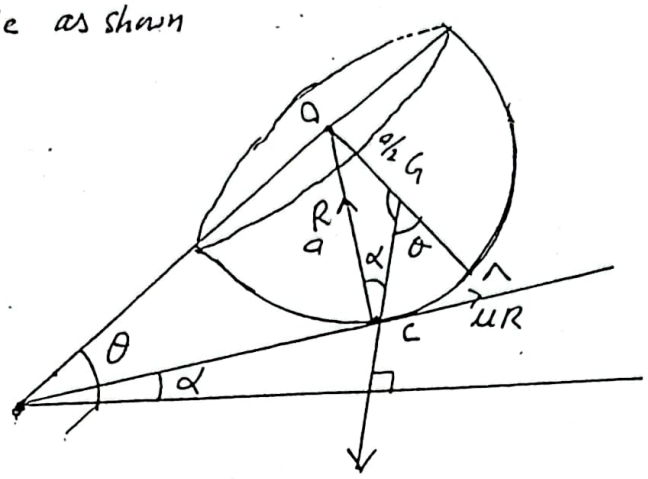
In equilibrium the max value of inclination α is λ

$\Rightarrow$ maximum value of θ is $\sin^{-1}(2\sin\lambda)$

$\Rightarrow \theta$ can not be greater than $\sin^{-1}(2\sin\lambda)$

Alternate Method

Fig can be made as shown



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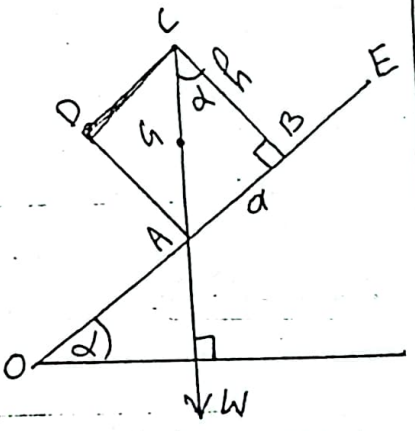
Example # A uniform rectangular block of height h whose base is a square of side (a), rests on a rough horizontal plane. The plane is gradually tilted about a line parallel to two edges of the base. Show that the block will slide or topple over according as $a \gtrless \mu h$, where μ is the co-efficient of friction.



Sol # Let α be the inclination of plane to horizontal when the block is on the point of toppling over (i.e in limiting equilibrium) Since the boloch falls or stands according as vertical through G meets the inclined plane within and outside the base of block, therefore vertical line Through G, C.g of the block must fall within the base.

$\therefore$ GA is vertical where A is the middle point of the lower edge of square base.

$\because$ CB $\perp$ OE & CW $\perp$ horizontal

$\therefore \angle ACB = \alpha$

$$\tan\alpha = \frac{AB}{BC} = \frac{a}{h}$$

Let θ be the inclination of plane when the block is at the point of sliding over. Then

$$\theta = \lambda$$

$$\Rightarrow \tan\theta = \tan\lambda = \mu$$

The block will topple over when $\theta > \alpha$ and will slide over when $\theta < \alpha$. i.e block will slide or topple over if

$$\theta \leqslant \alpha$$

If $\tan\theta \leqslant \tan\alpha$

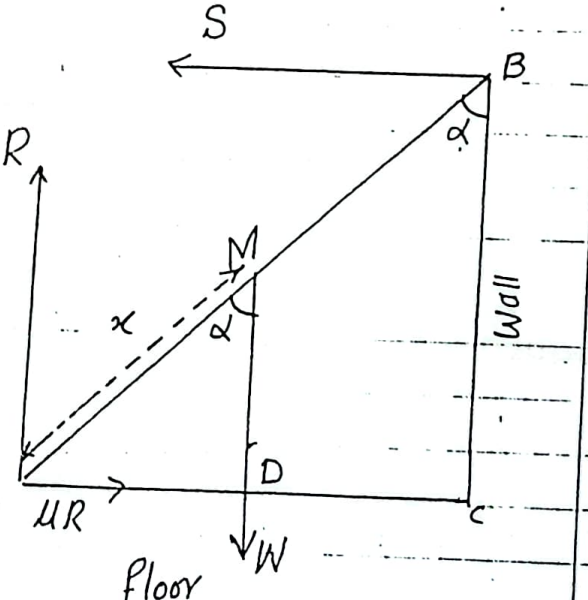
If $\mu \leqslant \frac{a}{h}$

if $a \geq \mu h$.

EXERCISE # 5

Q:1.# A light ladder is supported on a rough floor and leans against a smooth wall. How far up the ladder can a man climb without slipping taking place?

Sol # Let $AB = l$ be the light ladder inclined at an angle α to the vertical wall. R, S are normal reactions of floor and wall respectively. Suppose a man of weight W climbs upto point M when the ladder becomes at the point of slipping



Resolving the forces horizontally and vertically.

$$S = \mu R \longrightarrow (1)$$

$$R = W \longrightarrow (2)$$

Putting R in (1)

$$S = \mu W \longrightarrow (3)$$

Taking moments about A, we have

$$S \cdot BC = W \cdot AD$$

$$S \cdot l\cos\alpha = W \times \sin\alpha$$

putting $S = \mu W$ from (3)

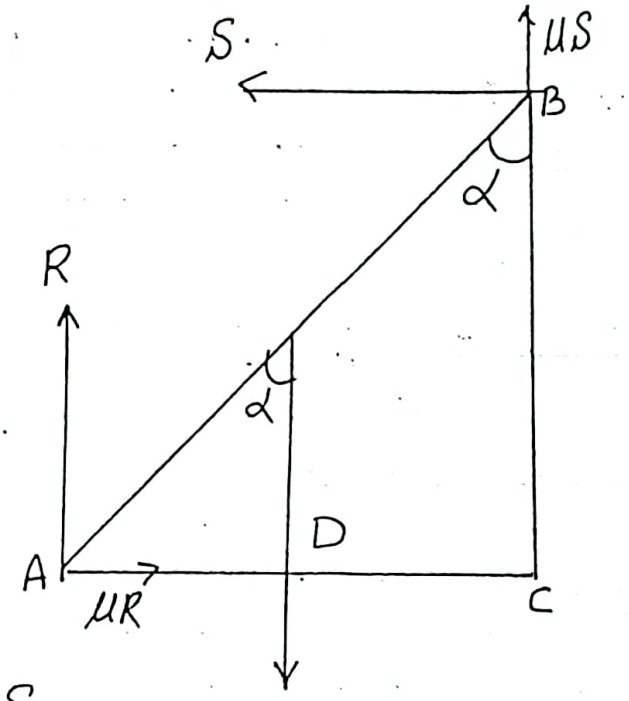
$$\mu W \cdot l\cos\alpha = W \times \sin\alpha$$

$$\mu l\cos\alpha = x \sin\alpha$$

$$x = \mu l \frac{\cos\alpha}{\sin\alpha} = \mu l \cot\alpha$$

Q:2 # Repeat Q:1 when both wall and floor are rough having the same co-efficient of friction μ

Sol # Let the man goes upto point M when the rod is at the point of slipping.



Resolving horizontally & vertically.

$$S = \mu R \longrightarrow (1)$$

$$\mu S + R = W \longrightarrow (2)$$

From (2)

$$R = W - \mu S$$

putting in (1)

$$S = \mu(W - \mu S)$$

$$S = \mu W - \mu^2 S \Rightarrow S(1+\mu^2) = \mu W$$

$$S = \mu W / 1+\mu^2 \longrightarrow (3)$$

Taking moment about point A.

$$S \cdot BC + \mu S \cdot AC = W \cdot AD$$

$$S \cdot l\cos\alpha + \mu S \cdot l\sin\alpha = W \cdot AM\sin\alpha$$

$$AM = \frac{Sl(\cos\alpha + \mu\sin\alpha)}{W\sin\alpha}$$

Putting $S = \frac{\mu W}{1+\mu^2}$

$$AM = \frac{\mu W l}{1+\mu^2}\frac{(\cos\alpha + \mu\sin\alpha)}{W\sin\alpha}$$

$$= \frac{\mu l}{1+\mu^2}(\mu + \cot\alpha)$$

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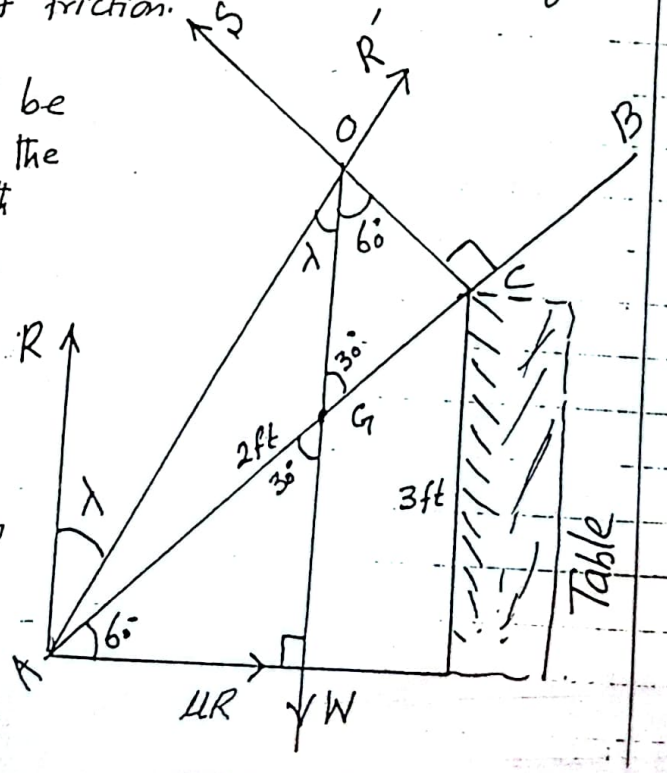
Q:3 # A rod 4 ft long rests on a rough floor against the smooth edge of a table of height 3ft. If the rod is on the point of slipping when inclined at an angle of 60° to the horizontal, find Co-efficient of friction.

Sol # let AB = 4ft be the rod and C be the point of contact with table. Since the edge of table is smooth, there will be only normal reaction S at C. Since the rod is in limiting equilibrium under the action of three forces R, S, W, these forces must be



concurrent. Suppose these forces meet at point O

$AB = 4ft \qquad AG = GB = 2ft$

$\overline{AC} = 3\,Cosec\,60°$

$= 2\sqrt{3}$

$GC = \overline{AC} - \overline{AG} = 2\sqrt{3} - 2$

$\angle AOG = \lambda$ (Alternate angles on lines of action of R & W) made by R'

$\angle COG = 60°$ (Angles between ⊥ars of AB and horizontal)

By (m,n) theorem, we have

$(\overline{AG} + \overline{GC})\,Cot30° = \overline{AG}\,Cot\lambda - \overline{GC}\,Cot60°$

$(2 + 2\sqrt{3} - 2)\,Cot30° = \frac{2}{\tan\lambda} - (2\sqrt{3} - 2)\cdot\frac{1}{\sqrt{3}}$

$2\sqrt{3}\cdot\sqrt{3} = \frac{2}{\mu} - (2\sqrt{3} - 2)\cdot\frac{1}{\sqrt{3}}$

$6 = \frac{2}{\mu} - 2 + \frac{2}{\sqrt{3}}$

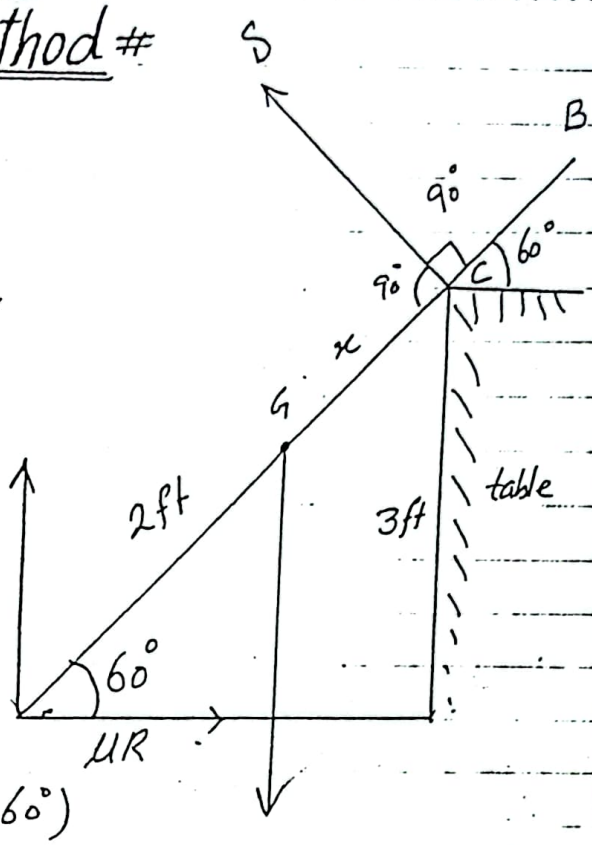
$\frac{2}{\mu} = 8 - \frac{2}{\sqrt{3}} = \frac{8\sqrt{3} - 2}{\sqrt{3}}$

$\frac{\mu}{2} = \frac{\sqrt{3}}{8\sqrt{3} - 2}$

$\mu = \frac{2\sqrt{3}}{2(4\sqrt{3} - 1)} = \frac{\sqrt{3}}{4\sqrt{3} - 1} = \frac{1.732}{5.928}$

$= .292.$

Alternate Method



Resolving the forces horizontally and vertically

$S \sin(90°+60°) + R = W$

$S\cos 60° + R = W$

$R = W - S\cos 60°$ $\longrightarrow$ ①

$S\cos(60+90°) = \mu R$

$S \sin 60° = \mu R \longrightarrow$ ②

putting ① in ②

$S \cdot \frac{\sqrt{3}}{2} = \mu(W - S\cos 60°)$

$= \mu(W - S \cdot \frac{1}{2})$

$= \mu W - \frac{\mu S}{2}$

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$\mu W = \frac{\sqrt{3}}{2} S + \frac{\mu S}{2}$

$2\mu W = (\sqrt{3} + \mu) S \Rightarrow S = \frac{2\mu W}{\sqrt{3}+\mu}$ $\longrightarrow$ ③

Also: $\frac{3}{2+x} = \sin 60° = \frac{\sqrt{3}}{2}$

$\frac{2+x}{3} = \frac{2}{\sqrt{3}}$

$2+x = \frac{6}{\sqrt{3}}$

Taking moments about point A

$W \cdot 2\cos 60° = S(2+x)$

$2W \cdot \frac{1}{2} = S \cdot \frac{6}{\sqrt{3}}$ $\left(\because 2+x = \frac{6}{\sqrt{3}}\right)$

$W = \frac{S\,6}{\sqrt{3}}$

Putting $S = \frac{2\mu W}{\sqrt{3}+\mu}$

$$W = \frac{2\mu W}{\sqrt{3}+\mu} \cdot \frac{6}{\sqrt{3}}$$

$$1 = \frac{12\mu}{3+\sqrt{3}\mu}$$

$$3+\sqrt{3}\mu = 12\mu.$$

$$3 = \mu(12-\sqrt{3})$$

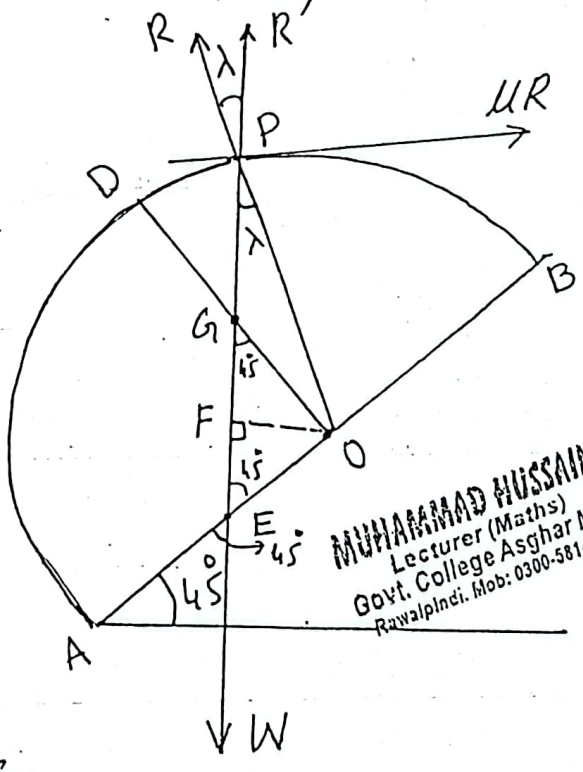
$$\mu = \frac{3}{12-\sqrt{3}} = 0.292.$$

Sol # Q:4 # A uniform semi-circular wire hangs on a rough peg, the line joining its extremities making an angle 45° with the horizontal. If it is just on the point of slipping, find the co-efficient of friction between the wire and peg.

Sol # Let P be the peg. If OD is symmetry radius of the semi-circular wire, then C.g G is at distance

$$OG = \frac{2a}{\pi}$$

Where (a) is the radius of circular wire.



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Forces acting on the semi-circular wire are normal reaction R along the radius through the peg, force of friction μR along the tangent at P and W through c.g of wire. If R' is the resultant reaction at (peg) wire, then it is equal to W and collinear with W because wire is in equilibrium

$m\angle GPO = \lambda$ (Vertical angle)

In $\triangle OGE$

$OG = OE$ (opposite sides of Congruent angles)

$\frac{2a}{\pi} = OE$

By applying law of Sines to $\triangle OEP$

$$\frac{OE}{\sin\lambda} = \frac{OP}{\sin 45^\circ}$$

$$\frac{\frac{2a}{\pi}}{\sin\lambda} = \frac{a}{\frac{1}{\sqrt{2}}}$$

$$\sin\lambda = \frac{2}{\pi\sqrt{2}} = \frac{\sqrt{2}}{\pi}$$

$$\mu = \tan\lambda = \frac{\sin\lambda}{\cos\lambda} = \frac{\sin\lambda}{\sqrt{1-\sin^2\lambda}}$$

$$= \frac{\sqrt{2}/\pi}{\sqrt{1-\frac{2}{\pi^2}}} = \frac{\sqrt{2}/\pi}{\frac{\sqrt{\pi^2-2}}{\pi}}$$

$$\mu = \frac{\sqrt{2}}{\sqrt{\pi^2-2}} = 0.504$$

Alternate Method

Resolving the forces along OP

$$R = W\cos\lambda \quad \rightarrow (1)$$

Taking moment about O.

$W \cdot OF = \mu R \cdot OP$

$W \cdot OG \sin 45^\circ = \mu R \cdot a$

$W \cdot \frac{2a}{\pi} \cdot \frac{1}{\sqrt{2}} = \mu R \cdot a$

$W \cdot \frac{\sqrt{2}}{\pi} = \mu R \quad \rightarrow (2)$

putting (1) in (2)

$$\frac{W\cdot\sqrt{2}}{\pi} = \mu\cdot W\cos\lambda$$

$$\frac{\sqrt{2}}{\pi\mu} = \cos\lambda$$

$$\frac{1}{\cos\lambda} = \frac{\pi\mu}{\sqrt{2}}$$

$$\sec\lambda = \frac{\pi\mu}{\sqrt{2}}$$

$$\sec^2\lambda = \frac{\pi\mu^2}{2}$$

$$1+\tan^2\lambda = \frac{\pi\mu^2}{2}$$

$$1+\mu^2 = \frac{\pi\mu^2}{2}$$

$$\mu^2\left(\frac{\pi^2-2}{2}\right) = +1$$

$$\mu^2 = \frac{2}{\pi^2-2}$$

$$\mu = \sqrt{\frac{2}{\pi^2-2}} = 0.504$$

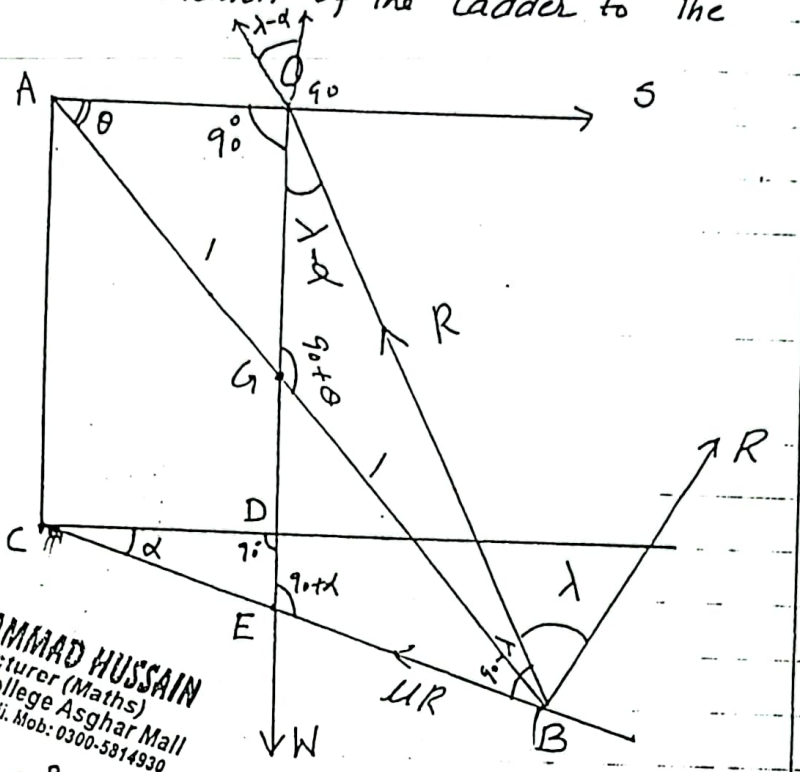
Q:-5 # One end of the uniform ladder of weight W, rests against a smooth wall and the other end on rough ground, which slopes down from the wall at an angle α to the horizon. Find the inclination of the ladder to the horizontal when it is at the point of sliding and show that the reaction of the wall is Then $W\tan(\lambda-\alpha)$, where λ is the angle of friction.

Sol # Let $AB = 2a$ be the ladder and R, S be the normal reaction at A & B. Let G be the centre of gravity of the ladder. Now the system is in limiting equilibrium under the action of three forces

(i) Weight W at G (ii) Reaction S at A

(iii) Reaction R at B.

Therefore those forces are concurrent at point O. Let θ be the inclination of the ladder to the horizontal.



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$m\angle AOG = 90°$

$m\angle OGB = 90° + \theta$ (External angle of Triangle OAG)

$m\angle OBE = 90 - \lambda$

$m\angle BEO = 90 + \alpha$ (External angle of Triangle DEC)

In ΔBOE

$m\angle BOE = 180° - (90 - \lambda + 90 + \alpha)$

$= \lambda - \alpha$

$AG : GB = 1 : 1$

Applying (m,n) theorem

$$(1+1)\cot(90+\theta) = 1\cdot\cot 90° - 1\cdot\cot(\lambda-\alpha)$$

$$-2\tan\theta = 0 - \cot(\lambda-\alpha)$$

$$\tan\theta = \frac{1}{2}\cot(\lambda-\alpha)$$

$$\theta = \tan^{-1}\left[\frac{\cot(\lambda-\alpha)}{2}\right]$$

Thus inclination with horizon $= \cot^{-1}[2\tan(\lambda-\alpha)]$

$= \text{Tan}^{-1}\left[\frac{\tan(\lambda-\alpha)}{2}\right]$

Also applying the Lamis Theorem

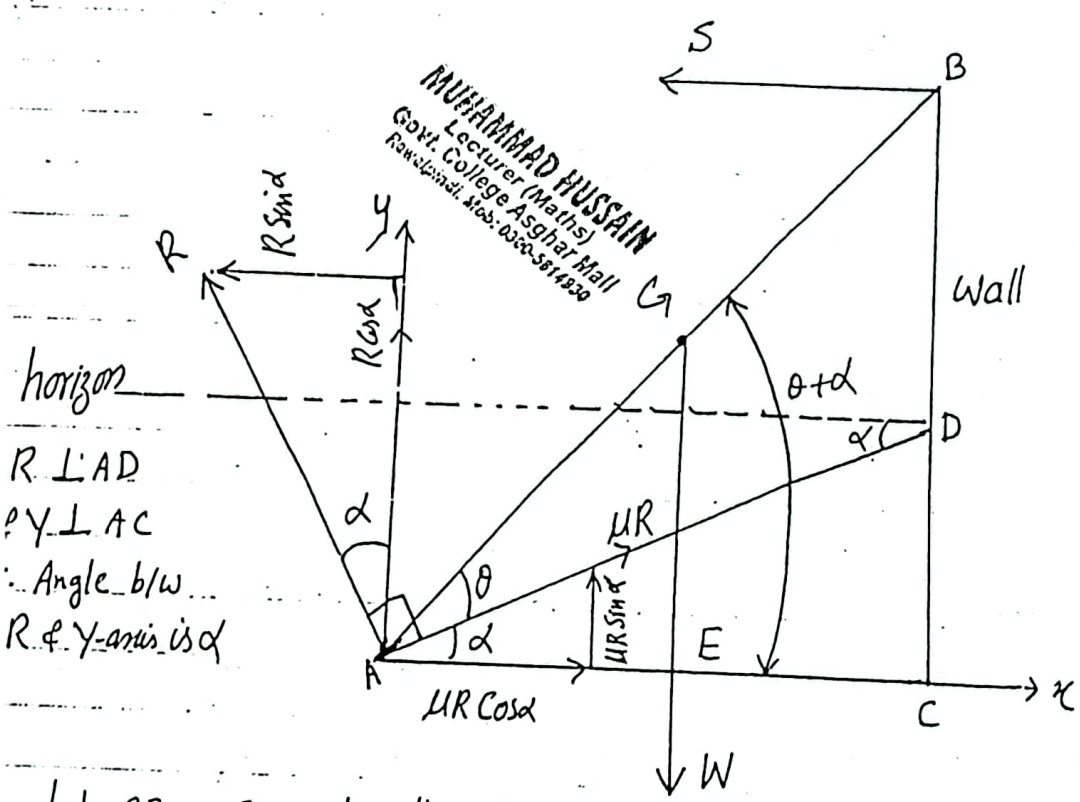
$$\frac{S}{\sin(\lambda-\alpha)} = \frac{W}{\sin(90^\circ+\lambda-\alpha)}$$

$$\frac{S}{\sin(\lambda-\alpha)} = \frac{W}{\cos(\lambda-\alpha)}$$

$$S = W\frac{\sin(\lambda-\alpha)}{\cos(\lambda-\alpha)}$$

$$S = W\tan(\lambda-\alpha)$$

Method-II



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Let AB = 2a be the ladder and R, S be the normal reactions at A & B. Let θ be the

angle of with ground, then angle of inclination of rod with horizon is $\theta + \alpha$

Resolving the forces horizontally and vertically.

$$S + R\sin\alpha = \mu R\cos\alpha \longrightarrow (1)$$

$$R\cos\alpha + \mu R\sin\alpha = W \longrightarrow (2)$$

from (2).

$$R = \frac{W}{\cos\alpha + \mu\sin\alpha}$$

pulting in (1)

$$S = R(\mu\cos\alpha - \sin\alpha)$$

$$= \frac{W}{\cos\alpha + \mu\sin\alpha}(\mu\cos\alpha - \sin\alpha)$$

$$= \frac{W(\tan\lambda\cos\alpha - \sin\alpha)}{\cos\alpha + \tan\lambda\sin\alpha}$$

$$= W\left(\frac{\sin\lambda\cos\alpha - \cos\lambda\sin\alpha}{\cos\lambda\cos\alpha + \sin\lambda\sin\alpha}\right)$$

$$= W\cdot\frac{\sin(\lambda-\alpha)}{\cos(\lambda-\alpha)}$$

$$\boxed{S = W\tan(\lambda-\alpha)}$$

Taking moments about A

$$S\cdot BC = W\cdot AE$$

$$S\cdot 2a\sin(\alpha+\theta) = W\,a\cos(\alpha+\theta)$$

$$\cot(\alpha+\theta) = \frac{2S}{W}$$

Putting value of $S = W\tan(\lambda-\alpha)$

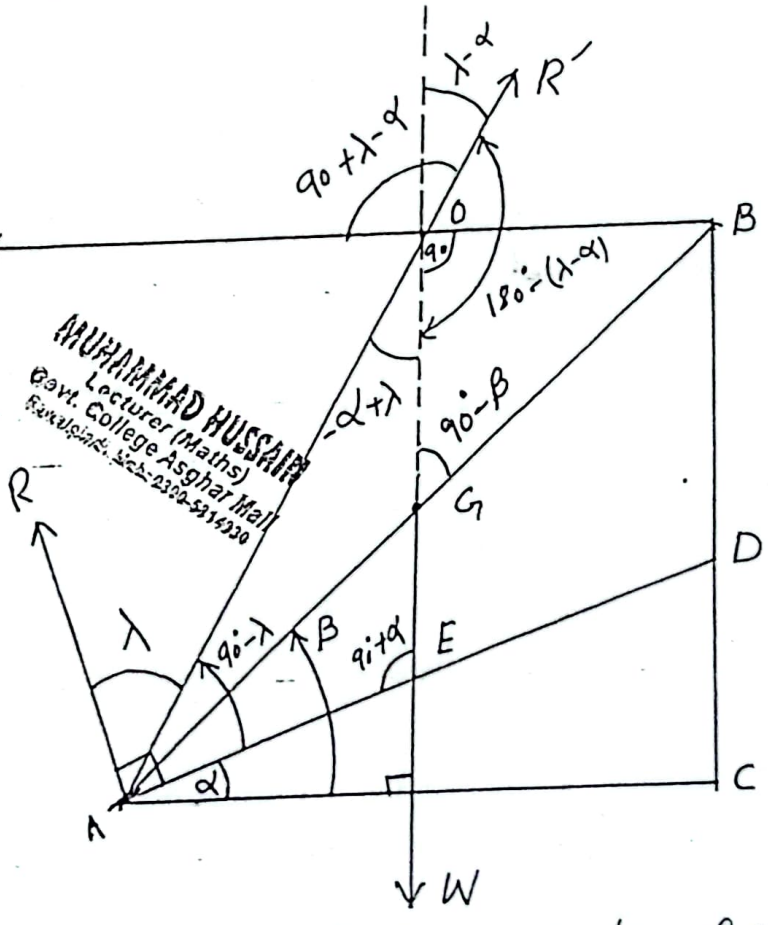
$$\cot(\alpha+\theta) = \frac{2}{W} \cdot W\tan(\lambda-\alpha)$$

$$\cot(\alpha+\theta) = 2\tan(\lambda-\alpha)$$

$$\alpha+\theta = \cot^{-1}[2\tan(\lambda-\alpha)]$$

Thus angle with horizon $= \cot^{-1}[2\tan(\lambda-\alpha)]$

Method-III



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Let AB be the ladder with G as centre of gravity, R' be the resultant reaction of the ground and S be the normal reaction of the wall. Now the system

is in equilibriam under the action of three forces R, S and W. These forces are concurrent at point O. Let β be the inclination of the ladder to the horizontal.

$m\angle BOG = 90°$

$m\angle OEA = 90 + \alpha$ (External angle of triangle)

$m\angle OAD = m\angle RAD - m\angle RAO$

$= 90° - \lambda$

In. ΔAOE

$m\angle AOG = 180° - (90° + \alpha + 90° - \lambda)$

$= \lambda - \alpha$

By (m,n) theorem ΔOAB

$(1+1)\cot(90° - \beta) = 1 \cdot \cot(\lambda - \alpha) - 1 \cdot \cot 90°$

$2\tan\beta = \cot(\lambda - \alpha)$.

$\therefore \tan\beta = \frac{1}{2}\cot(\lambda - \alpha)$

$\therefore \beta = \text{Tan}^{-1}\left[\frac{1}{2}\cot(\lambda - \alpha)\right]$

By Lamis theorem

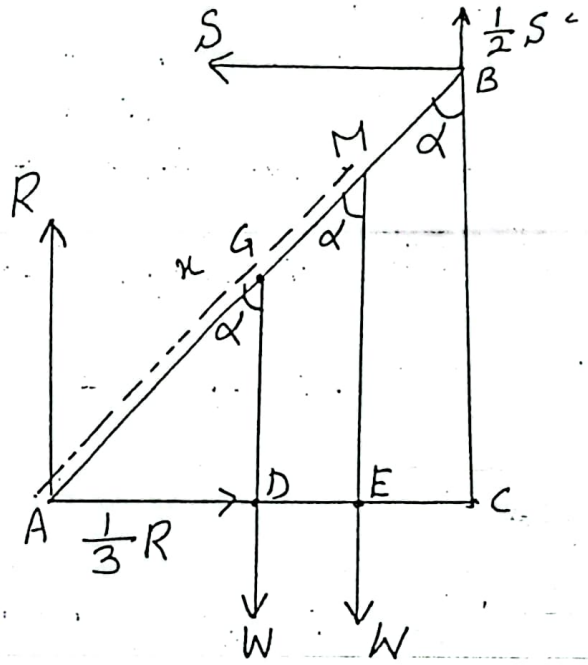
$$\frac{S}{\sin[180 - (\lambda - \alpha)]} = \frac{W}{\sin[90° + \lambda + \alpha]}$$

$$\frac{S}{\sin(\lambda - \alpha)} = \frac{W}{\cos(\lambda - \alpha)}$$

$$S = W\tan(\lambda - \alpha)$$

Q:6 # The upper end of a uniform ladder rests against a rough vertical wall and the lower end on a rough horizontal plane, the co-efficient of friction in both cases being $\frac{1}{3}$. Prove that if the inclination of the ladder to the vertical is $\tan^{-1}(\frac{1}{2})$, a weight equal to that of the ladder cannot be attached to it at a point, more than $\frac{9}{10}$ of the distance from the foot of it without destroying the equilibrium.

Sol # Let $AB = 2a$ be the rod and W be its weight. Suppose a weight W is attached at point M and the Ladder is limiting equilibrium. Let α be the inclination of ladder with wall. Then $\alpha = \tan^{-1}(\frac{1}{2})$



$\Rightarrow \tan\alpha = \frac{1}{2}$

Resolving the forces horizontally and vertically.

$$S = \frac{1}{3}R$$

$$\Rightarrow R = 3S \longrightarrow (1)$$

$$R + \frac{1}{3}S = 2W \longrightarrow (2)$$

putting value of R in (2)

$$3S + \frac{1}{3}S = 2W$$

$$\frac{10}{3}S = 2W \Rightarrow S = \frac{6}{10}W = \frac{3}{5}W$$

Taking moments about A

$$S \cdot BC + \frac{1}{3} S \cdot AC = W \cdot AE + W \cdot AD$$

$$S : 2a\cos\alpha + \frac{1}{3} S : 2a\sin\alpha = W \cdot x \sin\alpha + W a \sin\alpha$$

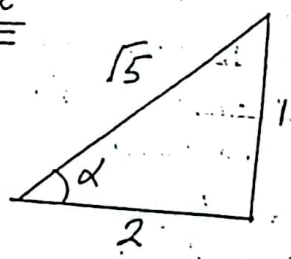
Putting $S = \frac{3}{5} W$

$$\frac{3}{5} W \cdot 2a\cos\alpha + \frac{1}{3} \cdot \frac{3}{5} \cdot 2a \sin\alpha = Wx\sin\alpha + Wa\sin\alpha$$

$$\frac{6}{5} a\cos\alpha + \frac{2}{5} a \sin\alpha = x\sin\alpha + a\sin\alpha$$

$$\because \alpha = \tan^{-1}(1/2) \Rightarrow \tan\alpha = \frac{1}{2}$$

$$\sin\alpha = 1/\sqrt{5} \quad \cos\alpha = \frac{2}{\sqrt{5}}$$



using these values in above.

$$\frac{6a}{5} \cdot \frac{2}{\sqrt{5}} + \frac{2a}{5} \cdot \frac{1}{\sqrt{5}} = x \cdot \frac{1}{\sqrt{5}} + \frac{a}{\sqrt{5}}$$

$$\frac{12a}{5} + \frac{2a}{5} = x + a$$

$$\frac{14}{5} a = x + a$$

$$x = \frac{14}{5} a - a = \frac{9}{5} a$$

$$= \frac{9}{10} (2a)$$

$$= \frac{9}{10} (\text{length of the ladder})$$

Hence a weight equal to the weight of ladder can not be attached to it at a point more than 9/10 of the distance from the foot of it without destroying equilibrium.

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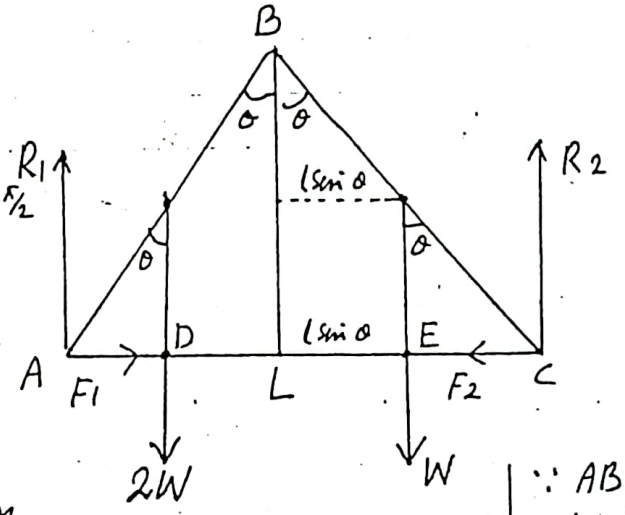
Q:7 # Two uniform beams AB, BC, of equal length are freely jointed at B and rest in equilibrium in a vertical plane with ends A and C on a rough horizontal plane (with the ends A and C). If the weight of AB is twice that of BC, show there can not be Limiting friction both at A and C and that if there is Limiting friction at either of these points, it is at C. Find also the co-efficient of friction if the greatest angle that rods can make with each other is a right angle.

Sol #

Let $m\angle ABC = 2\theta < \pi/2$ and $AB = BC = 2l$.

Let R_1, R_2 be the normal reactions at A & C respectively.

Suppose the friction is limiting at both ends A & C



Since the system is in equilibrium, therfore

$$F_1 = F_2 \longrightarrow (1)$$

$$R_1 + R_2 = 3W \longrightarrow (2)$$

Taking moments about point A.

$$R_2 \cdot AC = W \cdot AE + 2W \cdot AD$$

$$R_2 \cdot 4l\sin\theta = W \cdot 3l\sin\theta + 2W \cdot l\sin\theta$$

$$4R_2 = 3W + 2W$$

$$= 5W$$

$$R_2 = \frac{5W}{4} \longrightarrow (3)$$

from (2) & (3)

$$R_1 = 3W - R_2 = 3W - \frac{5W}{4} = \frac{7W}{4}$$

$\because AB = BC$

$\therefore$ Lar BL of this isosceles triangle will bisect the vertex angle 2θ

$\therefore m\angle ABL = m\angle CBL = \theta$

$AC = AL + LC$

$= 2l\sin\theta + 2l\sin\theta$

$= 4l\sin\theta$

$F_1 = \mu R_1 = \frac{7\mu W}{4}$

$F_2 = \mu R_2 = \frac{5\mu W}{4}$

$\Rightarrow \quad F_1 \neq F_2$

Which is a contradiction because $F_1 = F_2$. Hence it can not limiting friction at both at A & C simultaneously

Obviously friction force will reach its max value $\frac{5}{4}\mu W$ at point C before it reaches $\frac{7}{4}\mu W$.

Therefore if it is limiting friction at either point it will be at C because if it is at A, then equilibrium will be destroyed at point C.

When $2\theta = \pi/2$, Then it is the greatest angle that the rod can make with each other while being in equilibrium. Therefore the point C will be on the point of slipping but A will not. In this case

$$F_2 = \frac{5\mu W}{4}$$

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But $F_1 = F_2 = \frac{5\mu W}{4}$

Now taking moment about point B

$F_2 BL + W\frac{1}{2}BL = R_2 CL$. For Rod BC

$$\left(\frac{5}{4}\mu W + \frac{1}{2}W\right) 2l\cos 45^\circ = R_2\, 2l \sin 45^\circ$$

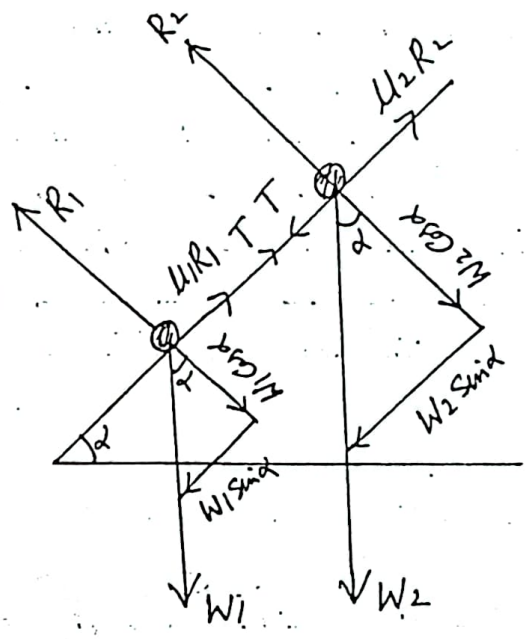
$$5/4\,\mu W + 1/2\,W = \frac{5}{4}W \qquad \because R_2 = \frac{5}{4}W \text{ from } (3)$$

$$5/4\,\mu = 5/4 - 1/2 = 3/4$$

$$\boxed{\mu = 3/5} \text{ As Required.}$$

Q:8 # Two bodies weights W_1, W_2 are placed on an inclined plane and are connected by light string which coincides with a line of greatest slope of plane. If the co-efficients of friction between the bodies and the plane be respectively μ_1, μ_2 find the inclination of the plane when both the bodies are on the point of motion, it being assumed that the smoother body below the other [Ans $\tan^{-1}\left(\frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}\right)$]

Sol # Let R_1, R_2 be the normal reactions and α be the inclination of the plane to the horizon when the bodies are on the point of motion downwards. Let T be the tension in the string



Equilibrium of W_1

Resolving the forces along and perpendicular to the plane.

$$R_1 = W_1 \cos\alpha \longrightarrow (1)$$

$$T + \mu_1 R_1 = W_1 \sin\alpha \longrightarrow (2)$$

putting value of R_1 from (1) into (2)

$$T = W_1 \sin\alpha - \mu_1 W_1 \cos\alpha \longrightarrow (3)$$

Equilibrium of W_2

Resolving the forces along and perpendicular to the plane

$$R_2 = W_2 \cos\alpha \longrightarrow (4)$$

$$\therefore \mu_2 R_2 = T + W_2 \sin\alpha \longrightarrow (5)$$

Putting R_2 from (4) into (5)

$$T = \mu_2 W_2 \cos\alpha - W_2 \sin\alpha \longrightarrow (6)$$

Equating (3) and (6) we have

$$W_1 \sin\alpha - \mu_1 W_1 \cos\alpha = \mu_2 W_2 \cos\alpha - W_2 \sin\alpha$$

$$(W_1 + W_2)\sin\alpha = (\mu_1 W_1 + \mu_2 W_2)\cos\alpha$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$$

$$\tan\alpha = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$$

$$\alpha = \tan^{-1}\left(\frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}\right) \text{ as Required.}$$

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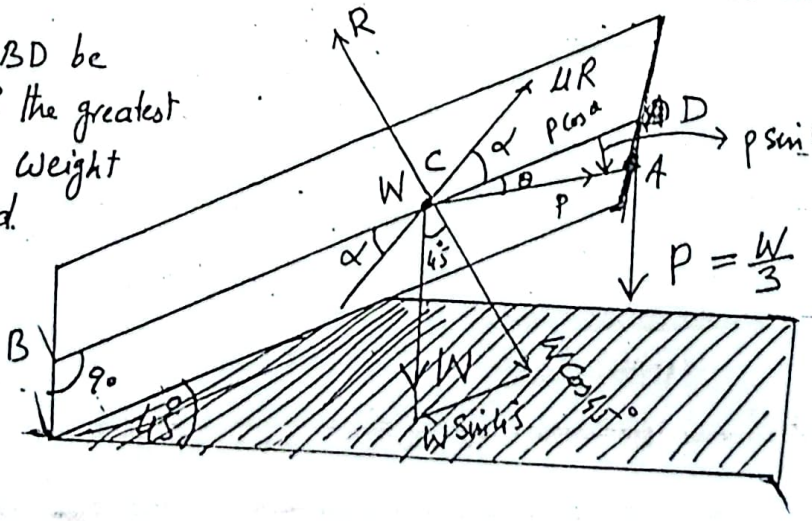
Q:9 # A weight W is laid upon a rough plane ($\mu = \frac{1}{\sqrt{3}}$) inclined at an angle 45° to the horizon and is connected by a string passing through a smooth ring A at the top of the plane with a weight P hanging vertically. If $W = 3P$, shows if θ be the greatest possible inclination of AW to the line of greatest slope in the plane, then

$$\cos\theta = \frac{2\sqrt{2}}{3}$$

Find the direction in which W would commence to move.

Sol #

Let BD be the line of the greatest slope and weight W is placed at point C



Let α be the angle with the line of the greatest slope of the direction in which W would commence to move. The weight W acts vertically downwards

Resolving normal to the plane or BD.

$$R = W\cos 45^\circ = \frac{W}{\sqrt{2}} \longrightarrow (1)$$

Resolving the forces along the plane or BD

$$P\cos\theta + \mu R\cos\alpha = W\sin 45^\circ$$

putting $P = \frac{W}{3}$ & $\mu = \frac{1}{\sqrt{3}}$, $R = \frac{W}{\sqrt{2}}$ from (1)

$$\frac{W}{3}\cos\theta + \frac{1}{\sqrt{3}}\cdot\frac{W}{\sqrt{2}}\cos\alpha = \frac{W}{\sqrt{2}}$$

$$\frac{1}{3}\cos\theta + \frac{1}{\sqrt{6}}\cos\alpha = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{6}}\cos\alpha = \frac{1}{\sqrt{2}} - \frac{1}{3}\cos\theta$$

$$\cos\alpha = \sqrt{3} - \frac{\sqrt{6}}{3}\cos\theta$$

$$\cos\alpha = \sqrt{3} - \sqrt{\tfrac{2}{3}}\cos\theta \longrightarrow (2)$$

Resolving the forces perpendicular to the ~~plane~~ BD

$$P\sin\theta = \mu R\sin\alpha$$

putting $P = \frac{W}{3}$ $\mu = \frac{1}{\sqrt{3}}$, $R = \frac{W}{\sqrt{2}}$

$$\frac{W}{3}\sin\theta = \frac{1}{\sqrt{3}}\cdot\frac{W}{\sqrt{2}}\sin\alpha$$

$$\frac{1}{3}\sin\theta = \frac{1}{\sqrt{6}}\sin\alpha$$

$$\sin\alpha = \frac{\sqrt{6}}{3}\sin\theta = \sqrt{\frac{2}{3}}\sin\theta \longrightarrow (3)$$

Squaring and adding (2) & (3)

$$\cos^2\alpha + \sin^2\alpha = 3 + \tfrac{2}{3}\cos^2\theta - 2\sqrt{2}\cos\theta + \tfrac{2}{3}\sin^2\theta$$

$$1 = 3 + \frac{2}{3}(\cos^2\theta + \sin^2\theta) - 2\sqrt{2}\cos\theta$$

$$= 3 + \frac{2}{3} - 2\sqrt{2}\cos\theta$$

$$= \frac{11}{3} - 2\sqrt{2}\cos\theta$$

$$\Rightarrow 2\sqrt{2}\cos\theta = \frac{11}{3} - 1 = \frac{8}{3}$$

$$\cos\theta = \frac{8}{3\cdot 2\sqrt{2}} = \frac{4}{3\sqrt{2}} = \frac{2\cdot 2}{3\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{3}$$

Putting value of $\cos\theta$ in ②

$$\cos\alpha = \sqrt{3} - \sqrt{\frac{2}{3}}\cdot\frac{2\sqrt{2}}{3}$$

$$= \sqrt{3} - \frac{4}{3\sqrt{3}} = \frac{9-4}{3\sqrt{3}} = \frac{5}{3\sqrt{3}}$$

$$= \frac{5\sqrt{3}}{9}$$

$$\alpha = \cos^{-1}\left(\frac{5\sqrt{3}}{9}\right)$$

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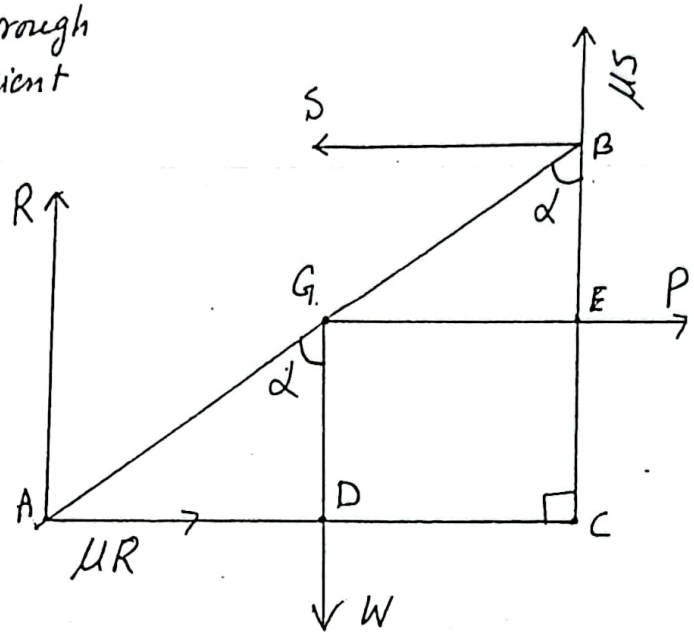
Q: 10 # A uniform rod of weight W is placed with its lower end on a rough horizontal floor and its upper end against and equally rough vertical wall. The rod makes an angle α with the wall and is just prevented from slipping down by a horizontal force P applied at its middle point. Prove that

$$P = W\tan(\alpha - 2\lambda)$$

where λ is the angle of friction and $\lambda < \frac{1}{2}\alpha$.

Sol # Let $AB = 2l$ be the rod of weight W and inclined at α with vertical wall. P is the horizontal force applied at the middle point of AB. Let R, S be the normal reactions of ground and wall.

Since the ground and wall are equally rough, therefore co-efficient of friction for the bolt is same



Resolving the forces horizontally and vertically

$$P + \mu R = S \longrightarrow (1)$$

$$R + \mu S = W \longrightarrow (2)$$

From (1)

$$P = S - \mu R \longrightarrow (3)$$

From (2) $W = R + \mu S \longrightarrow (4)$

Dividing (4) by (3)

$$P/W = \frac{S - \mu R}{R + \mu S} = \frac{1 - \mu R/S}{R/S + \mu} \longrightarrow (5)$$

Taking moment about point G.

$$R \cdot AD = \mu R \cdot GD + S \cdot BE + \mu S \cdot GE$$

$$R \cdot l \sin\alpha = \mu R \cdot l \cos\alpha + S \cdot l \cos\alpha + \mu S \cdot l \sin\alpha$$

$$R(\sin\alpha - \mu \cos\alpha) = S(\cos\alpha + \mu \sin\alpha)$$

$$R/S = \frac{\cos\alpha + \mu \sin\alpha}{\sin\alpha - \mu \cos\alpha}$$

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$$R/S = \frac{\cos\alpha + \frac{\sin\lambda}{\cos\lambda}\sin\alpha}{\sin\alpha - \frac{\sin\lambda}{\cos\lambda}\cos\alpha}$$

$$= \frac{\cos\alpha\cos\lambda + \sin\lambda\sin\alpha}{\sin\alpha\cos\lambda - \sin\lambda\cos\alpha}$$

$$R/S = \frac{\cos(\alpha-\lambda)}{\sin(\alpha-\lambda)}$$

Putting value of R/S in (5)

$$\frac{P}{W} = \frac{1 - \mu\frac{\cos(\alpha-\lambda)}{\sin(\alpha-\lambda)}}{\frac{\cos(\alpha-\lambda)}{\sin(\alpha-\lambda)} + \mu}$$

$$= \frac{1 - \frac{\sin\lambda}{\cos\lambda}\cdot\frac{\cos(\alpha-\lambda)}{\sin(\alpha-\lambda)}}{\frac{\cos(\alpha-\lambda)}{\sin(\alpha-\lambda)} + \frac{\sin\lambda}{\cos\lambda}}$$

$$= \frac{\sin(\alpha-\lambda)\cos\lambda - \cos(\alpha-\lambda)\sin\lambda}{\cos(\alpha-\lambda)\cos\lambda + \sin(\alpha-\lambda)\sin\lambda}$$

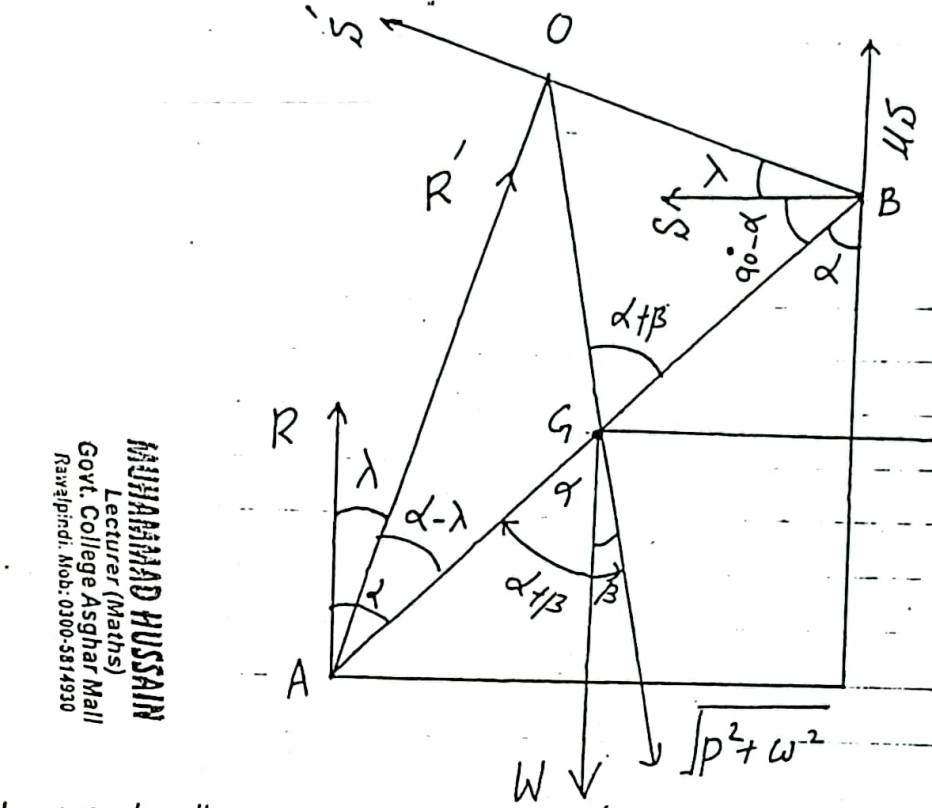
$$= \frac{\sin(\alpha-\lambda-\lambda)}{\cos(\alpha-\lambda-\lambda)} = \frac{\sin(\alpha-2\lambda)}{\cos(\alpha-2\lambda)}$$

$$P = W\tan(\alpha-2\lambda)$$

Since the force P must act along GE, it must be +ve i.e

$$\alpha - 2\lambda > 0 \Rightarrow \lambda < \alpha/2.$$

Alternate Method



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Let AB be the rod and S', R' be the resultant reactions at B and A. Resultant of forces P, W is $\sqrt{P^2+W^2}$ which acts at point G. Now the system is in equilibriam under the forces S', R' and $\sqrt{P^2+W^2}$. Therefore forces are Concurrent at point O.

By (m-n) theorem in ΔOAB

$$(1+1)\cot(\alpha+\beta) = 1\cot(\alpha-\lambda) - 1\cdot\cot[90^\circ-\alpha+\lambda]$$

$$2\cot(\alpha+\beta) = \cot(\alpha-\lambda) - \cot[90^\circ-(\alpha-\lambda)]$$

$$2\cot(\alpha+\beta) = \cot(\alpha-\lambda) - \tan(\alpha-\lambda)$$

$$2\cot(\alpha+\beta) = \frac{1}{\tan(\alpha-\lambda)} - \tan(\alpha-\lambda)$$

$$= \frac{1-\tan^2(\alpha-\lambda)}{\tan(\alpha-\lambda)}$$

$$\frac{2}{\tan(\alpha+\beta)} = \frac{1-\tan^2(\alpha-\lambda)}{\tan(\alpha-\lambda)}$$

$$\tan(\alpha+\beta) = \frac{2\tan(\alpha-\lambda)}{1-\tan^2(\alpha-\lambda)}$$

$$= \tan(\alpha-\lambda+\alpha-\lambda)$$

$$= \tan(2\alpha-2\lambda)$$

$$\alpha+\beta = 2\alpha-2\lambda$$

$$\beta = \alpha-2\lambda$$

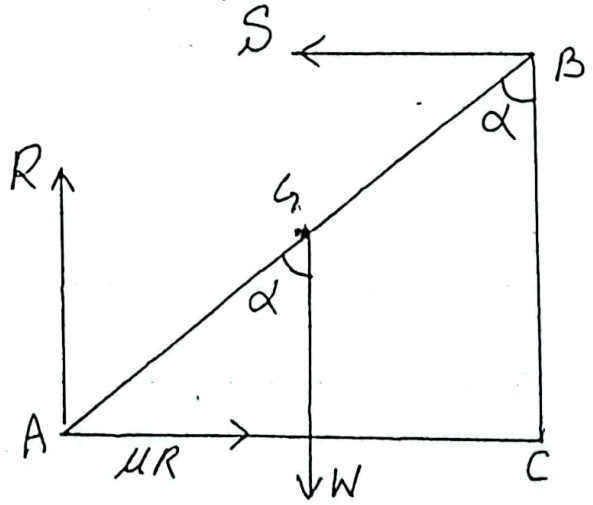
$$\tan\beta = \tan(\alpha-2\lambda)$$

$$P/W = \tan(\alpha-2\lambda)$$

$$P = W\tan(\alpha-2\lambda)$$

Q:11# A uniform ladder rests in limiting equilibrium with one end on a rough horizontal plane and the other against a smooth vertical wall. A man ascends the ladder. Show that he can not go more than half way up.

Sol # Let $AB = 2a$ be the ladder with angle α as its inclination to the vertical in the limiting equilibrium position.



Equilibrium Without Man.

Resolving the forces horizontally and vertically

$$S = \mu R \longrightarrow (1)$$
$$R = W \longrightarrow (2)$$

Putting value of R in (1)

$$S = \mu W \longrightarrow (3)$$

Taking moments about point A.

$$S \cdot BC = W \cdot AD$$
$$S \cdot 2a\cos\alpha = W \cdot a\sin\alpha$$

Putting $S = \mu W$

$$\mu W \cdot 2a\cos\alpha = W \cdot a\sin\alpha$$

$$2\mu = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

$$\alpha = \tan^{-1}(2\mu)$$

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OR

$m\angle AOG = \lambda$ $m\angle BOG = 90°$
$m\angle OGB = \alpha$

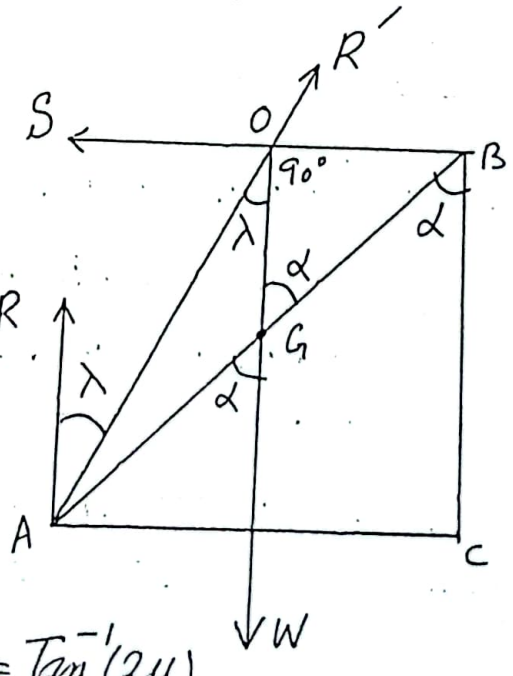
By (m,n) Theorem to $\triangle OAB$

$(1+1)\cot\alpha = 1\cdot\cot\lambda - 1\cdot\cot 90°$

$$\frac{2}{\tan\alpha} = \frac{1}{\tan\lambda} - 0$$

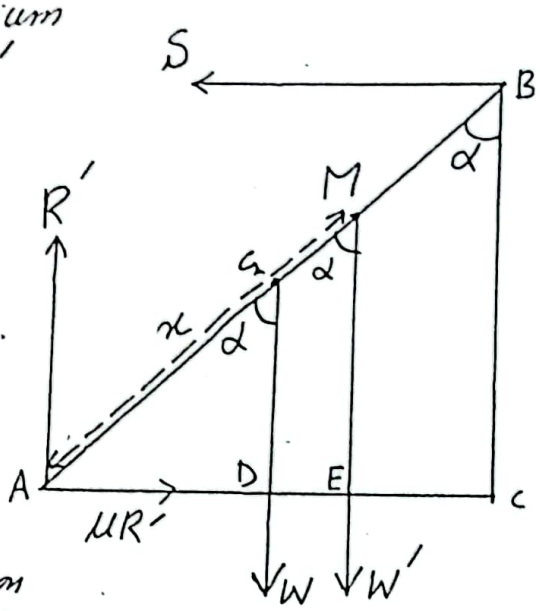
$$\frac{2}{\tan\alpha} = \frac{1}{\mu}$$

$$\tan\alpha = 2\mu \Rightarrow \alpha = \tan^{-1}(2\mu)$$



Equilibrium With Man

When the man ascends the ladder which itself is in limiting equilibrium, the man's weight W' increases the normal reaction at A i.e $R' = R + W'$ and force of friction at A is $\mu(R+W') > \mu R$. The ladder comes in equilibrium due to increase in friction at A.



Let M be the position of man of wt W' when the ladder comes in limiting equilibrium.

Resolving the forces horizontally and vertically

$$R' = W + W' \longrightarrow (1)$$

$$S = \mu R' \longrightarrow (2)$$

Putting (1) in (2)

$$S = \mu(W+W') \longrightarrow (3)$$

Taking moments about point A

$$S \cdot BC = W \cdot AD + W' \cdot AE$$

$$S \cdot 2a\cos\alpha = W \cdot a\sin\alpha + W' \cdot x \cdot \sin\alpha$$

putting $S = \mu(W+W')$

$$2a\mu(W+W')\cot\alpha = Wa + W'x$$

putting $\cot\alpha = \frac{1}{2\mu}$ $(\because \tan\alpha = 2\mu)$

$$2a\mu(W+W')\cdot\frac{1}{2\mu} = Wa + W'x$$

$$aW + aW' = Wa + W'x$$

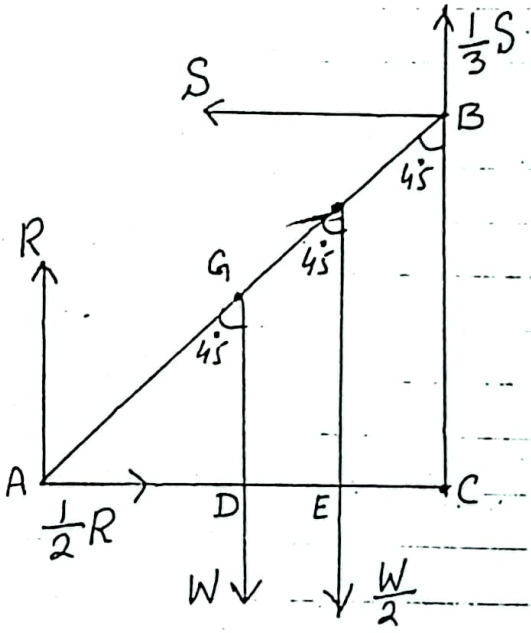
$$x = a = \frac{1}{2}(2a)$$

$\Rightarrow$ man can not go more than half way up.

Q:12 # A uniform ladders, of length 70 ft, rests against a vertical wall with which it makes an angle of 45°, the co-efficients of friction between ladder and the wall and ground respectively being $\frac{1}{3}$, and $\frac{1}{2}$. If a man, whose weight is one half that of the ladder, ascends the ladder, how high will he be when the ladder slips.

If a boy now stands on the bottom rung of the ladder what must be his least weight so that he may go to the top of the ladder? (Ans 50 ft $\frac{1}{4}$ of the wt of ladder)

Sol # Let AB = 70 ft be the ladder of weight W and a man of wt $\frac{W}{2}$ is at distance AM = x from the foot when the rod is in limiting equilibrium.



Resolving the forces horizontally and vertically

$$S = \frac{1}{2}R$$

$$R = 2S \longrightarrow (1)$$

$$R + \frac{1}{3}S = W + W/2 = \frac{3W}{2}$$

putting R from (1)

$$2S + \frac{1}{3}S = \frac{3W}{2}$$

$$\Rightarrow S = \frac{9}{14}W \longrightarrow (2)$$

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Taking moments about A

$$W\,AD + \frac{W}{2}\cdot AE = S\,BC + \frac{1}{3}S\cdot AC$$

$$W(35\sin 45^\circ) + \frac{W}{2}(x\sin 45^\circ) = S(70\cos 45^\circ) + \frac{S}{3}(70\sin 45^\circ)$$

$$35W + \frac{x}{2}W = 70S + \frac{70S}{3}$$

$$W\left(35 + \frac{x}{2}\right) = \left(70 + \frac{70}{3}\right)S$$

putting $S = \frac{9W}{14}$

$$W\left(35 + \frac{x}{2}\right) = \left(70 + \frac{70}{3}\right)\cdot\frac{9W}{14}$$

$$35 + \frac{x}{2} = \overset{5}{70}\times\frac{9}{14} + \frac{\overset{5}{70}}{\underset{3}{3}}\times\frac{\overset{3}{9}}{14}$$

$$= 45 + 15$$

$$= 60.$$

$$x/2 = 60 - 35 = 25$$

$$x = 50\ ft$$

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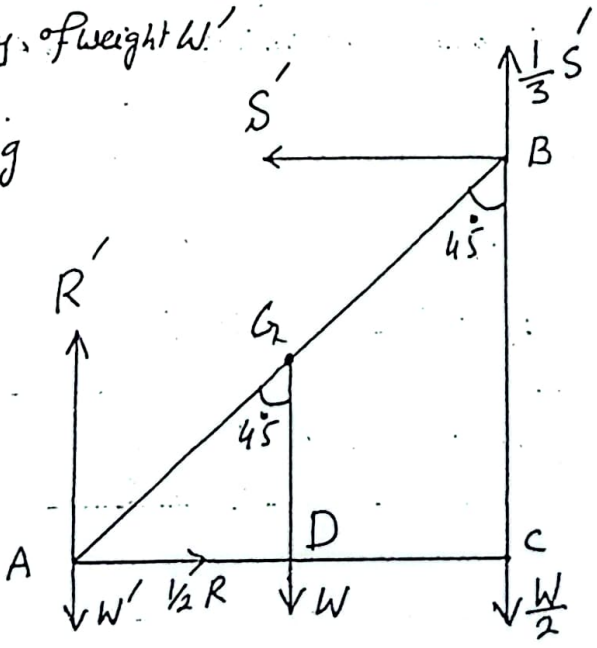
Case-II # Let a boy of weight W' stands on the bottom rung of the ladder so that the ladder is again limiting equilibrium when the man is at the top B.

Resolving the forces vertically and horizontally we have



$$S' = \frac{1}{2}R' \qquad \underline{S4}$$

$$R' = 2S' \longrightarrow (3)$$

$$R' + \frac{1}{3}S' = W' + \frac{3W}{2}$$

Putting $R' = 2S'$

$$2S' + \frac{1}{3}S' = W' + \frac{3W}{2}$$

$$\frac{7}{3}S' = W' + \frac{3}{2}W$$

$$S' = \frac{3}{7}\left(W' + \frac{3}{2}W\right) \longrightarrow (4)$$

Taking moments about A

$$W \cdot AD + \frac{W}{2}AC = S' \cdot BC + \frac{1}{3}S' \cdot AC$$

$$W \cdot 35\sin 45^\circ + \frac{W}{2}(70\sin 45^\circ) = S'(70\cos 45^\circ) + \frac{1}{3}S'(70\sin 45^\circ)$$

$$35W \cdot \frac{1}{\sqrt{2}} + 35W \cdot \frac{1}{\sqrt{2}} = \frac{70S'}{\sqrt{2}} + \frac{70}{3}S' \cdot \frac{1}{\sqrt{2}}$$

$$70W = \frac{280}{3}S'$$

Putting $S' = \frac{3}{7}(W' + \frac{3}{2}W)$

$$70W = \frac{280}{3} \cdot \frac{3}{7}(W' + \frac{3}{2}W)$$

$$70W = 40(W' + \frac{3}{2}W)$$

$$= 40W' + 60W$$

$$70W - 60W = 40W'$$

$$10W = 40W'$$

$$W' = \frac{1}{4}W = \frac{1}{4}(\text{weight of the ladder})$$

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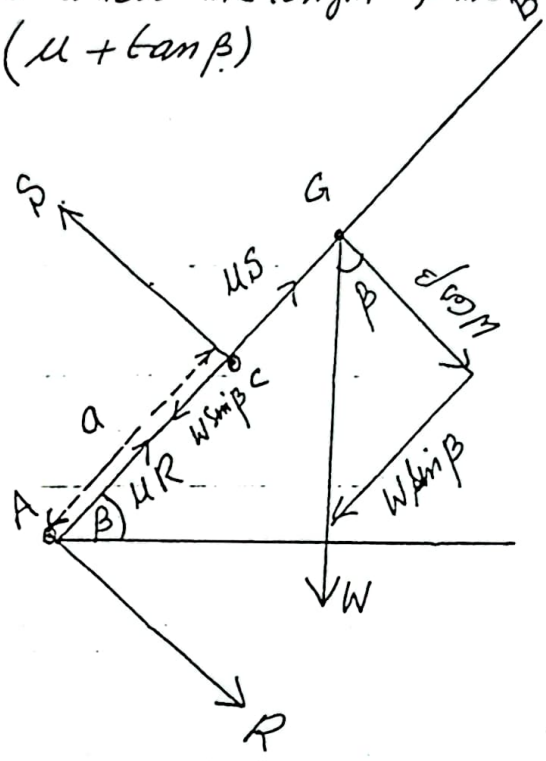
Q:13 # A thin uniform rod passes over one peg and under another, the co-efficient of friction between each peg and the rod μ. The distance between the pegs is a, and the straigh line joining them makes an angle β with the horizontal. Shows that equilibrium is not possible unless the length of the rod is greater than $\frac{a}{\mu}(\mu + \tan\beta)$

Sol #

Let $AB = 2l$ be the rod resting over the peg C and under the peg A where $AC = a$. The C.G of the rod preceedes peg C otherwise equilibrium is not possible.



Resolving the forces along and perpendicular to the rod.

$$S = R + W\cos\beta \longrightarrow (1)$$

$$\mu R + \mu S = W\sin\beta \longrightarrow (2)$$

From (1)

$$R = S - W\cos\beta$$

Puting value of R in (2)

$$\mu(S - W\cos\beta) + \mu S = W\sin\beta$$

$$2\mu S = \mu W\cos\beta + W\sin\beta$$

$$S = \frac{W}{2\mu}(\mu\cos\beta + \sin\beta) \longrightarrow (3)$$

Taking moments about A, we get

$$W\cos\beta \cdot AG = S\,AC$$

$$W\cos\beta \frac{AB}{2} = S\,a$$

$$AB = \frac{2a}{W\cos\beta} \cdot S$$

Putting value of S from (3)

$$AB = \frac{2a}{W\cos\beta} \cdot \frac{W}{2\mu}(\mu\cos\beta + \sin\beta)$$

$$= \frac{a}{\mu\cos\beta}(\mu\cos\beta + \sin\beta)$$

$$= \frac{a}{\mu}\left(\frac{\mu\cos\beta}{\cos\beta} + \frac{\sin\beta}{\cos\beta}\right)$$

$$= \frac{a}{\mu}(\mu + \tan\beta)$$

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Hence the equilibrium is possible if the length of rod is at least $\frac{a}{\mu}(\tan\beta + \mu)$

Q:14 # A uniform rod, of length $2a$ and weight W rests with its middle point upon a rough horizontal cylinder whose axis is perpendicular to the rod. Show that the greatest weight that can be attached to one end of the rod, without sliding it off the cylinder, is $\frac{b\lambda}{a - b\lambda}$

Where b is the radius of the cylinder and λ is the angle of friction

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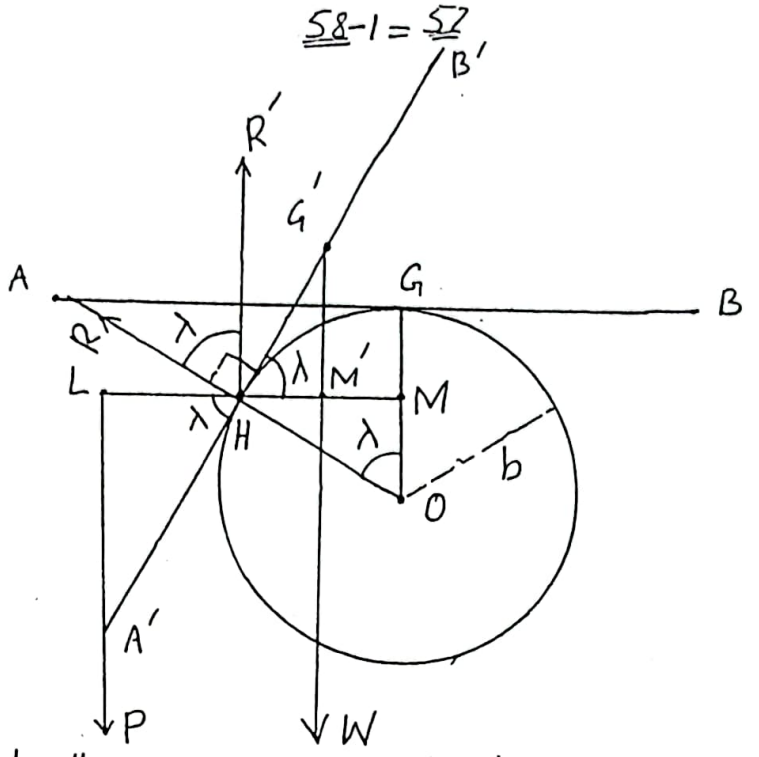
58-1 = 57

Sol #



Let AGB be the rod resting in the horizontal position with middle point G in touch with the cylinder. Let A'G'B' be the limiting equilibrium position when a weight P is attached at end A. Since the rod is in equilibrium under the action of three forces W, P, R' and W, P act vertically downward (i.e parallel), therefore R' must also be vertically upward at H., and must balance the resultant of P & W.

$m\angle GOH = \lambda$ (Corresponding angles made by transversal OR on parallel lines OG & HR')

$m\angle A'HL = m\angle MHG' = \lambda$ (Vertical angles lines LM & A'B')

Since W at G' and P at A' balance about the point H, therefore we take moments about H.

$$P \cdot LH = W \cdot HM'$$

$$P \cdot A'H \cos\lambda = W \cdot HG' \cos\lambda$$

$$P \cdot A'H = W \cdot HG'$$

$$P(A'G' - HG') = W \cdot HG' \longrightarrow (1)$$

Since the rotation of the rod is very very small, therefore we have.

59-1 = 58

$\widehat{HG} \approx \overline{HG'} = b\lambda$

using this value in ①

$P(a - b\lambda) = Wb\lambda$

$AB = 2a$

$A'G' = a$

$P = \frac{Wb\lambda}{a - b\lambda}$ As required.

Geometry # If two parallel lines are cut by a transversal, each pair of the corresponding angles are congruent.

Q:15 # A solid cylinder rests on a rough horizontal plane with one of its flat ends on the plane and is acted on by a horizontal force the centre of its upper end. If this force is just sufficient to move the solid, show that it will slide and not topple over if the co-efficient of friction be less than the ratio of radius of the base of the cylinder to its height.

Sol #

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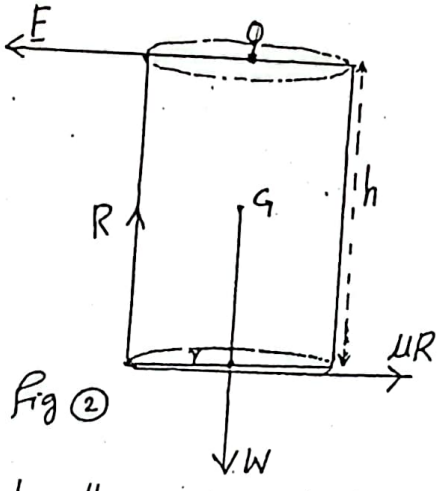

Fig ②

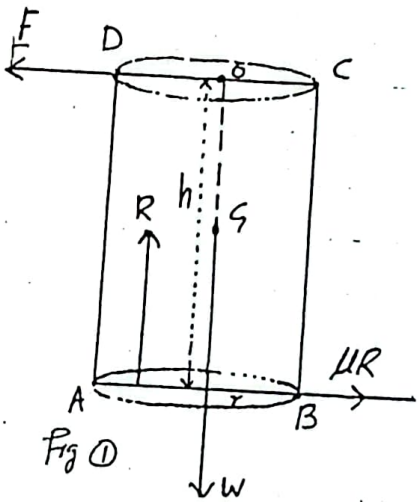

Fig ①

Let r be the radius, h be the height and G be the C.G of the cylinder. Suppose F be the horizontal force applied at the centre of upper end and is just sufficient to move the solid. So in the limiting equilibrium let normal reaction R acts at some point of its area of contact. and friction force μR in the direction of AB as shown in fig ①.

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60 - 1 = 59

Resolving the forces horizontally and vertically, we get

$$R = W \longrightarrow (1)$$

and $$F = \mu R \longrightarrow (2)$$

Clearly the block will slide if

$$F > \mu R = \mu W$$

$$F > \mu W \longrightarrow (3)$$

If the block is about to topple over and is touch the plane at A, the reaction must act at A.

In this case the two couples (F at O, R at A) and (R at A and W at G) will act at the cylinder. Let G_1 and G_2 be moments of these couples. The cylinder will not topple over

$$\text{If } G_1 < G_2$$

$$\text{If } Fh < Wk$$

$$\text{If } F < \frac{Wk}{h} \longrightarrow (4)$$

By (3) and (4) the block will slide and not topple over

if $\mu W < F < \frac{Wk}{h}$.

or $\mu W < \frac{Wk}{h}$

or $\mu < \frac{k}{h}$ As required

The End of the Exercise.

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